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Nonlinear Dynamics of a Rigid Block on a Rigid Base

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NONLINEAR: DYNAMICS: RIGID BLOCK

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The planar rocking of a prismatic rectangular rigid block about either of its corners is considered. The problem of homoclinic intersections of the stable and unstable manifolds of the perturbed separatrix is addressed to and the corresponding Melnikov functions are derived. Inclusion of the vertical forcing in the Hamiltonian permits the construction of a three-dimensional separatrix. The corresponding modified Melnikov function of Wiggins for homoclinic intersections is derived. Further, the 1-period symmetric orbits are predicted analytically using the method of averaging and compared with the simulation results. The stability boundary for such orbits is also established.

Introduction

This paper investigates the response of a rigid rectangular block on a rigid base, in planar rocking motion about its corners. This problem has been studied by several investigators in the past. Housner (1963) drew attention to the fact that rocking of blocks is of relevance in earthquake engineering. Spanos and Koh (1984) observed that for slender blocks the equation of motion will be piecewise linear. This property was exploited to construct a symmetric periodic solution. They also conducted detailed numerical work to arrive at a boundary in the parameter space beyond which the block would topple. Hogan (1989, 1990, 1992) in a series of papers has extensively studied the rocking response of blocks. He considered prismatic blocks on a rigid base under sinusoidal excitation. Initially the impacts with the base were assumed to cause no energy dissipation. A variety of periodic orbits and also chaotic motion were found possible. Further improvements, to include damping, have been handled numerically. The piecewise linearity of slender blocks has been used by Hogan also to construct periodic solutions. Tso and Wong (1989a) have used this property to predict the existence and stability of harmonic and subharmonic responses. They followed this study (Wong and Tso, 1989b) by an experimental investigation wherein the existence of three-period and quasi-periodic orbits were demonstrated. Yim and Lin (1991) have extended the study of rocking of slender objects by constructing the Melnikov function. This helps in identifying regions in the parametric space where, chaotic response may be possible. Through numerical studies, they further exhibit the possibility of chaotic behavior in rigid blocks. Shenton and Jones (1991a, 1991b) have considered the periodic slide-rock motion of rigid blocks. Augustin and Sinopoli (1992) have derived the equation of motion of a rocking block including static and kinetic dry friction. They have also delineated the region where rocking is possible as functions of static friction and shape of the block. Recently Lipscombe and Pellegrino (1993) investigated through theory and experiments the effect of bouncing of short blocks after each impact with the base. Their study further highlights the extreme sensitivity of rocking response to geometric imperfections and errors in the relevant parameters occuring in the equation of motion.

The present study investigates the full nonlinear system without the assumption of slenderness and consequent piecewise linear property. The Melnikov functions for homoclinic intersections of two different types are derived. The effect of vertical base excitation is also included. It is shown that the modified Melnikov function of Wiggins (1988) can be constructed when harmonic vertical and lateral base excitations are simultaneously present. Further an approximate analytical solution is presented for a one period orbit, valid for short blocks also. Numerical results are obtained in the parameter space to illustrate the application of the analytical solutions.

Equation of Motion

The system under consideration is shown in Fig. 1. Under the action of base accelerations $u''(t)$ and $v''(t)$ the prismatic rectangular block rocks about the edge passing through the points O and O_1. Taking moments of the forces about a corner the equation of motion for the planar rotation θ can be shown to be

$$I\theta'' + I(1-\nu)\theta'|\theta'|\delta(\theta) + WR\sin(\alpha \operatorname{sgn}\theta - \theta) \times (1 + v''/g) + WR\cos(\alpha \operatorname{sgn}\theta - \theta)u''/g = 0. \quad (1)$$

Here the primes denote derivatives with respect to time, $\delta(\cdot)$ is the Dirac delta function and sgn $(\cdot)$ is the signum function. I is the moment of inertia of the block about a corner; ν is the coefficient of restitution, defined as the ratio of the angular velocities immediately after and before an impact; W is the weight of the block and R is the distance of the centroid of the cross section from a corner. The angle θ is taken to be positive when the block rotates about the corner O_1. The base excitations are taken as $u''(t)/g = u_m \sin(\lambda_h t)$ and $v''(t)/g = v_m \sin(\lambda_v t)$. Now introducing the dimensionless parameters $\epsilon_1 = (1-\nu)$, $\epsilon_2 = WR/I\nu_h^2$, $\epsilon_3 = u_m\epsilon_2$, $\epsilon_4 = v_m\epsilon_2$, $\Omega = \lambda_v/\lambda_h$ and changing the independent variable to $\tau = \lambda_h t/2\pi$ one gets

$$\ddot\theta + \epsilon_1\dot\theta|\dot\theta|\delta(\theta) + 4\pi^2\epsilon_2\sin(\alpha\operatorname{sgn}\theta - \theta) = -4\pi^2\epsilon_3\sin(2\pi\tau)\cos(\alpha\operatorname{sgn}\theta - \theta) - 4\pi^2\epsilon_4\sin(2\pi\Omega\tau)\sin(\alpha\operatorname{sgn}\theta - \theta). \quad (2)$$

The above equation is valid only if rocking gets initiated. The condition for initiation of rocking is $\epsilon_3 \geq \epsilon_2(1 - \epsilon_4/\epsilon_2) \times \tan\alpha$. In case the horizontal forcing is absent, the frequency ratio Ω in the above equation is undefined and hence minor modifications are needed. The equation of motion for a block driven by only vertical excitation is taken as

$$\ddot\theta + \epsilon_1\dot\theta|\dot\theta|\delta(\theta) + 4\pi^2\epsilon_5\sin(\alpha\operatorname{sgn}\theta - \theta) = -4\pi^2\epsilon_6\sin(2\pi\tau)\sin(\alpha\operatorname{sgn}\theta - \theta). \quad (3)$$

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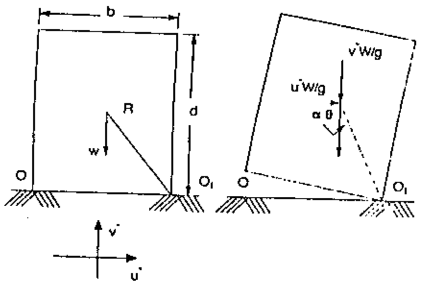

Fig. 1 Rocking rigid block

Here the new dimensionless parameters ϵ_5 and ϵ_6 are defined as $\epsilon_5 = WR/I\lambda_v^2$, $\epsilon_6 = v_m\epsilon_5$. In this case, in order that rocking gets initiated, the initial conditions should be nonzero and $\epsilon_4 > \epsilon_2$. All the parameters excepting ϵ_2 and ϵ_5 are taken to be small quantities. Further, the time variable τ is replaced by t for convenience.

Vertical Input as a Small Perturbation

Equation (2) which represents the more general case can be recast as

$$\dot{\theta}_1 = f_1$$
$$\dot{\theta}_2 = f_2 + g_2$$
$$\dot{\psi} = 1 \tag{4}$$

where

$$f_1 = \theta_2, f_2 = -4\pi^2\epsilon_2 \sin(\alpha \operatorname{sgn} \theta_1 - \theta_1)$$
$$g_2 = -\epsilon_1\theta_2|\theta_2|\delta(\theta_1) - 4\pi^2\epsilon_3 \sin(2\pi t)\cos(\alpha \operatorname{sgn} \theta_1 - \theta_1)$$
$$- 4\pi^2\epsilon_4 \sin(2\pi\Omega t)\sin(\alpha \operatorname{sgn} \theta_1 - \theta_1).$$

The Separatrix

The above equations constitute an autonomous Hamiltonian system if $\epsilon_1 = \epsilon_3 = \epsilon_4 = 0$, with

$$H(\theta_1, \theta_2) = 0.5\theta_2^2 + 4\pi^2\epsilon_2 \cos(\alpha \operatorname{sgn} \theta_1 - \theta_1). \tag{5}$$
$$\theta_2 = \dot{\theta}_1$$

The phase plane for such a system is shown in Fig. 2. The system has three singular points, namely a neutrally stable center at $(\theta_1, \theta_2) = (0, 0)$ and a pair of unstable saddle points at $(\theta_1, \theta_2) = (\pm\alpha, 0)$. The saddle points are connected by the separatrix S, which is the level curve corresponding to $H(\pm\alpha, 0) = H_0 = 4\pi^2\epsilon_2$. An explicit expression for S can be found by solving the differential equation

$$0.5\theta_2^2 + 4\pi^2\epsilon_2 \cos(\alpha \operatorname{sgn} \theta_1 - \theta_1) = 4\pi^2\epsilon_2 \tag{6}$$

with the initial condition $\theta_1(0) = 0$. The expressions for the homoclinic trajectories are

$$\theta_{1\pm}(t) = \pm\operatorname{sgn}(t)[\alpha - 4\tan^{-1}\{\exp(-2\pi\epsilon_2^{1/2}t \operatorname{sgn}(t)\tan(\alpha/4))\}] \tag{7}$$

$$\theta_{2\pm}(t) = \pm[1 + \tan^2(\alpha/4)\exp\{-4\pi\epsilon_2^{1/2}t \operatorname{sgn}(t)\}]^{-1} \times [8\pi\epsilon_2^{1/2}\tan(\alpha/4)\exp\{-2\pi\epsilon_2^{1/2}t \operatorname{sgn}(t)\}] \tag{8}$$

The separatrix S is given by

$$S = \{\theta_{1+}, \theta_{2+}\} \cup \{\theta_{1-}, \theta_{2-}\} \cup \{\alpha, 0\} \cup \{-\alpha, 0\} \tag{9}$$

where $(\alpha, 0)$ and $(-\alpha, 0)$ are the limit points for the trajectories $\theta_{1+}(t)$ and $\theta_{1-}(t)$, respectively, as $t \to \infty$. It may be noted that in the above derivations, the relations $\operatorname{sgn}(\theta_{1+}) = \operatorname{sgn}(t)$, $\operatorname{sgn}(\theta_{1-}) = -\operatorname{sgn}(t)$ have been used.

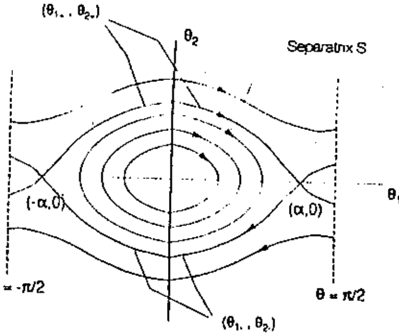

Fig. 2 Phase plane of Eq. (5)

Melnikov Function

For the present problem possibilities of homoclinic intersections of the stable and unstable manifolds exist. In such cases, simple zeros of the associated Melnikov function would indicate transversal intersections between the stable and unstable manifolds. This in turn would hint at the possibility of chaos. Here it may be pointed out that the functions, f_2 and g_2 are C^1 except at $\theta = 0$. Even for such a case, it can be shown that the perturbed homoclinic trajectories are ϵ-close to the unperturbed ones. Further, it may be geometrically demonstrated (Appendix) that transversal intersections between the stable and unstable manifolds are not possible whenever $\theta_1 = 0$. Therefore the transversality arguments (Wiggins, 1990) are not violated. Hence if q' denotes the $O(\epsilon)$ correction to the unperturbed homoclinic trajectory, q, then q' can be expressed piecewise continuously by the first variational equation

$$q'(t, t_0) = \mathbf{Df}\{q(t - t_0)\}q'(t, t_0) + \mathbf{g}\{q(t - t_0), t\}, t, t_0 \in (-\infty, 0) \cup (0, \infty) \tag{10}$$

where $\mathbf{f}$ and $\mathbf{g}$ are vector functions defined as

$$\mathbf{f} = \{f_1 \quad f_2\}^T \text{ and } \mathbf{g} = \{g_1 \quad g_2\}^T \tag{11}$$

and $\mathbf{D}$ stands for the Jacobian of the vector function $\mathbf{f}$ with respect to its argument q. It may be mentioned that $\theta_1 = t = t_0 = 0$ corresponds to a point of discontinuity where the first variational Eq. (10) is not meaningful.

Now, referring to Fig. 3, it may be argued that two types of transversal intersections are possible. In Type I, stable and

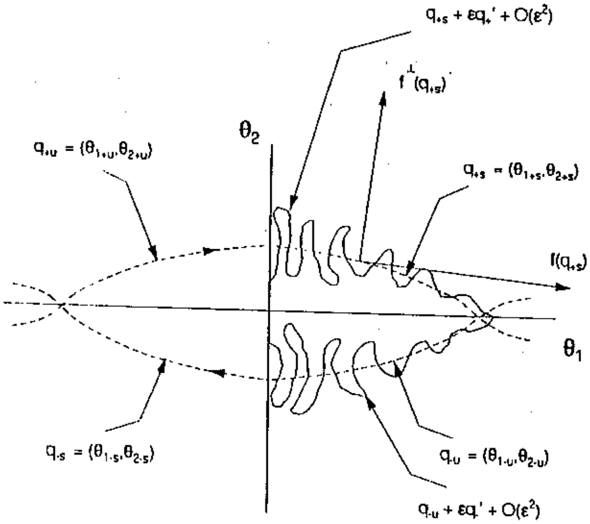

Fig. 3 Poincaré section of the perturbed phase plane of Eq. (4)

unstable manifolds of two different hyperbolic fixed points may intersect. Thus the existence of homoclinic points $p_1 \in' q_{+s} \cap' q_{+u}$ is investigated. On the other hand if $\epsilon_2 \ll 1$, i.e., if the maximum distance of separation of the unperturbed stable and unstable manifolds of the same fixed point is $O(\epsilon)$, then it is possible that homoclinic points of the type $p_2 \in' q_{+s} \cap' q_{-u}$ may exist even under a perturbative field of $O(\epsilon)$. This is referred to as homoclinic intersections of Type II.

Homoclinic Intersections—Type I

With reference to Eq. (4), the Melnikov function may be found from the well-known wedge product (Guckenheimer and Holmes, 1986)

$$M_{h1} = \int_{-\infty}^{\infty} \mathbf{f}\{\theta_{1+}(t), \theta_{2+}(t)\} \wedge \mathbf{g}\{\theta_{1+}(t), \theta_{2+}(t), (t + t_0)\}dt. \quad (12)$$

Here the subscript '$h1$' stands for homoclinic intersection of Type I. It is noted that the Melnikov function corresponding to $\{\theta_{1-}, \theta_{2-}\}$ would be identical to the above. Now expansion of the right-hand side of Eq. (12) followed by suitable simplification leads to

$$M_{h1} = -8\pi^2\epsilon_1\epsilon_2(1 - \cos\alpha) - 64I_u\pi^3\epsilon_2^{1/2}\epsilon_3 \tan(\alpha/4)\sin(2\pi t_0) - 64I_v\pi^3\epsilon_2^{1/2}\epsilon_4 \tan(\alpha/4)\cos(2\pi\Omega t_0) \quad (13)$$

where

$$I_u = \int_0^{\infty} \{1 + \tan^2(\alpha/4)\exp(-4\pi\epsilon_2^{1/2}t)\}^{-1}\exp(-2\pi\epsilon_2^{1/2}t) \times \cos(2\pi t)\cos[4\tan^{-1}\{\tan(\alpha/4)\exp(-2\pi\epsilon_2^{1/2}t)\}]dt$$

and

$$I_v = \int_0^{\infty} \{1 + \tan^2(\alpha/4)\exp(-4\pi\epsilon_2^{1/2}t)\}^{-1} \times \exp(-2\pi\epsilon_2^{1/2}t)\sin(2\pi\Omega t) \times \sin[4\tan^{-1}\{\tan(\alpha/4)\exp(-2\pi\epsilon_2^{1/2}t)\}]dt. \quad (14)$$

In deriving the above expressions for M_{h1} the properties of the separatrix, S, namely, $\theta_{2+}(t) = \theta_{2+}(-t)$, and $\theta_{1+}(t) = -\theta_{1+}(-t)$, have been used. Equation (13) contain I_u and I_v which are infinite integrals. An alternative expression for M_{h1} can be derived which is computationally more convenient by observing that $\theta_{2+}dt = d\theta_{1+}$. Use of this in Eq. (12) leads to

$$M_{h1} = -8\pi^2\epsilon_1\epsilon_2(1 - \cos\alpha) - 8\pi^2\epsilon_3 \sin 2\pi t_0 K_u + 8\pi^2\epsilon_4\cos(2\pi\Omega t_0)K_v \quad (15)$$

where

$$K_u = \int_0^{\alpha} \cos[(1/\epsilon_2^{1/2})\ln\{\tan(0.25(\alpha - \phi))/\tan(0.25\alpha)\}] \times \cos(\alpha - \phi)d\phi \quad (16)$$

and

$$K_v = \int_0^{\alpha} \sin[(\Omega/\epsilon_2^{1/2})\ln\{\tan(0.25(\alpha - \phi))/\tan(0.25\alpha)\}] \times \sin(\alpha - \phi)d\phi. \quad (17)$$

It is to be noted that the Melnikov function for the combined forcing as given by Eq. (15) is valid only for rational values of Ω. Only in this case, we can find a least common multiplier ψ_c of the time periods of horizontal and vertical excitations given, respectively, by $T_1 = 1.0$ and $T_2 = 1/\Omega$ such that $g_2(\theta_1, \theta_2, t) = g_2(\theta_1, \theta_2, t + \psi_c)$. In case the horizontal excitation is absent, the equation of motion is given by Eq. (3). Again the Melnikov function can be found as

$$M_{h1v} = -8\pi^2\epsilon_1\epsilon_5(1 - \cos\alpha) + 8\pi^2\epsilon_6\cos(2\pi t_0)L_v. \quad (18)$$

Here L_v is given by

$$L_v = \int_0^{\alpha} \sin[(1/\epsilon_2^{1/2})\ln\{\tan(0.25(\alpha - \phi))/\tan(0.25\alpha)\}] \times \sin(\alpha - \phi)d\phi \quad (19)$$

Homoclinic Intersection—Type II

It is here assumed that $\epsilon_2 \ll 1$. Referring to Fig. 3, the vector notation $q_+ = (q_{+s} \cup q_{+u}) = (\theta_{1+s}, \theta_{2+s}) \cup (\theta_{1+u}, \theta_{2+u})$ is introduced for the upper half of the unperturbed homoclinic orbit. A similar notation is valid for q_-, the lower half. The subscripts s and u stand for the stable and unstable manifolds. Clearly, the unperturbed homoclinic orbit is given by $q = (q_+ \cup q_-)$. Now a separation vector is defined between the stable and unstable manifolds of the perturbed fixed point as $d(t_0) = \{' q_{+s}(t_0) - ' q_{-u}(t_0)\}$. Here, t_0 is an arbitrary time when the Poincaré section as shown in Fig. 3 is chosen. The time-dependent distance function is

$$\Delta(t, t_0) = \mathbf{f}\{q_+'(t - t_0)\} \wedge \{q_+'(t, t_0) - q_-'(t, t_0)\} = \Delta_s(t, t_0) - \Delta_u(t, t_0) \quad (20)$$

so that the Melnikov function for homoclinic intersections would be given by

$$M_{h2}(t_0) = \Delta(t_0, t_0) = \int_{t_0}^{\infty} \Delta_s(t, t_0)dt - \int_{-\infty}^{t_0} \Delta_u(t, t_0)dt. \quad (21)$$

Here, q_+ and q_- are the $O(\epsilon)$ corrections to q_{+s} and q_{-u} given by the first variational equation (10). Now, following Guckenheimer and Holmes (1986), one can show that

$$\dot{\Delta}_s(t, t_0) = \mathbf{f}\{q_+(t - t_0)\} \wedge \mathbf{g}\{q_+(t - t_0), t\}. \quad (22)$$

Similarly, taking the time derivative of $\Delta_u(t, t_0)$ and using Eq. (10), one has

$$\dot{\Delta}_u(t, t_0) = \mathbf{D}f\{q_+(t - t_0)\}\mathbf{f}\{q_+(t - t_0)\} \wedge q_-'(t, t_0) + \mathbf{f}\{q_+(t - t_0)\} \wedge [\mathbf{D}\mathbf{f}\{q_-(t - t_0)\}q_-'(t, t_0) + \mathbf{g}\{q_-(t - t_0), t\}]. \quad (23)$$

It is noted here that $\Delta_u(-\infty, t_0) = 0$. Further, in the present problem,

$$f_1(\theta_1, \theta_2) = f_1(\theta_1) = \theta_2$$

and

$$\partial f_2(\theta_{1+}, \theta_{2+})/\partial\theta_{1+} = \partial f_2(\theta_{1-}, \theta_{2-})/\partial\theta_{1-}. \quad (24)$$

Expansion of the right side of Eq. (23), use of the above conditions, and finally substitution of the resulting expressions in Eq. (21) followed by a change of variable from t to $t + t_0$ leads to the following Melnikov function

$$M_{h2} = \int_{0+}^{\infty} \mathbf{f}\{q_+(t)\} \wedge \mathbf{g}\{q_+(t), (t + t_0)\}dt + \int_{-\infty}^{0-} \mathbf{f}\{q_+(t)\} \wedge \mathbf{g}\{q_-(t), (t + t_0)\}dt \quad (25)$$

for homoclinic intersections. Further simplification is possible by noting that along the unperturbed separatrix, $\theta_{1+}(t) = -\theta_{1-}(t)$ and $\theta_{2+}(t) = -\theta_{2-}(t)$. This leads to

$$M_{h2} = -8\pi^2\epsilon_1\epsilon_2(1 - \cos\alpha)$$

$$- 8\pi^2\epsilon_3 \sin(2\pi t_0)K_u - 8\pi^2\epsilon_4 \sin(2\pi\Omega t_0)K_{1v}. \quad (26),$$

Here

$$K_{1v} = \int_0^\alpha \cos[(\Omega/\epsilon_2^{1/2}) \ln\{\tan(0.25(\alpha - \phi))/\tan(0.25\alpha)\}]$$

$$\times \sin(\alpha - \phi)d\phi. \quad (27)$$

It is seen that for $\epsilon_4 = 0$, we have $M_{h1} = M_{h2}$. Hence the onset of chaos via Type I and Type II intersections follow the same pattern for blocks driven by only the horizontal acceleration. In case, $\epsilon_3 = 0$, one gets the following expression for the block driven by only a vertical excitation

$$M_{h2v} = -8\pi^2\epsilon_1\epsilon_5(1 - \cos\alpha) - 8\pi^2\epsilon_6 \sin(2\pi t_0)L_{1v} \quad (28)$$

where

$$L_{1v} = \int_0^\alpha \cos[(1/\epsilon_5^{1/2}) \ln\{\tan(0.25(\alpha - \phi))/\tan(0.25\alpha)\}]$$

$$\times \sin(\alpha - \phi)d\phi. \quad (29)$$

Vertical Input as Parametric Excitation

Wiggins (1988) has developed a method which is more general than that of Melnikov. Parametric excitations with large amplitudes but with small frequencies can be handled by this approach. For this purpose, Eq. (4) is rewritten as

$$\dot\theta_1 = f_1$$

$$\dot\theta_2 = f_2 + g_2$$

$$\dot z = \Omega$$

$$\dot\psi = 1. \quad (30)$$

Here

$$f_1(\theta_1, \theta_2) = \theta_2$$

$$f_2(\theta_1, \theta_2) = -4\pi^2\epsilon_2 \sin(\alpha \operatorname{sgn}\theta_1 - \theta_1)\{1 + (\epsilon_4/\epsilon_2)\sin(2\pi z)\}$$

$$g_2(\theta_1, \theta_2, \psi) = -\epsilon_1\theta_2|\theta_2|\delta(\theta_1) - 4\pi^2\epsilon_3 \sin(2\pi\psi)\cos(\alpha \operatorname{sgn}\theta_1 - \theta_1) \quad (31)$$

where z and ψ are modulo $1/\Omega$ and 1, respectively.

Three-Dimensional Separatrix

It is observed that the above system is Hamiltonian when $\epsilon_1 = \epsilon_3 = 0$. The time variable for such a system gets uncoupled from the rest of the equations. The hyperbolic fixed points for this case are $(-\alpha, 0, z)$ and $(\alpha, 0, z)$ for all $\epsilon_4 < \epsilon_2$ and for all z in the interval $(0, 1/\Omega]$. The corresponding phase space is illustrated in Fig. 4. The Hamiltonian is given by the energy functional

$$H(\theta_1, \theta_2, z) = 0.5\theta_2^2 + 4\pi^2\epsilon_2 \cos(\alpha \operatorname{sgn}\theta_1 - \theta_1) \times \{1 + (\epsilon_4/\epsilon_2)\sin(2\pi z)\}. \quad (32)$$

To obtain the expressions for the homoclinic trajectories the above equation is solved for $H_0(z) = 4\pi^2\epsilon_2\{1 + (\epsilon_4/\epsilon_2)\sin(2\pi z)\}$. This leads to

$$\theta_{1\pm}(t, z) = \pm\operatorname{sgn}(t)[\alpha - 4\tan^{-1}\{\exp(-2\pi\mu(z)t \operatorname{sgn}(t))\tan(\alpha/4)\}] \quad (33)$$

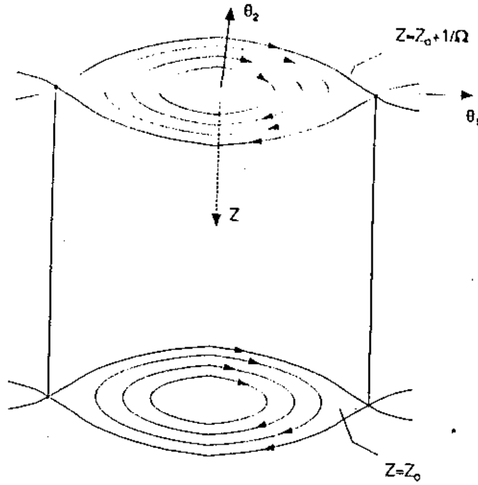

Fig. 4 Phase space for the system of Eqs. (30)

and

$$\theta_{2\pm}(t, z) = \pm\{1 + \tan^2(\alpha/4)\exp\{-4\pi\mu(z)t \operatorname{sgn}(t)\}\}^{-1} \times [8\pi\mu(z)\tan(\alpha/4)\exp\{-2\pi\mu(z)t \operatorname{sgn}(t)\}] \quad (34)$$

where $\mu(z) = [\epsilon_2\{1 + (\epsilon_4/\epsilon_2)\sin(2\pi z)\}]^{1/2}$.

The separatrix, S, for this case is given by

$$S = \{\theta_{1+}, \theta_{2+}, z\} \cup \{\theta_{1-}, \theta_{2-}, z\} \cup \{\alpha, 0, z\} \cup \{-\alpha, 0, z\}. \quad (35)$$

Wiggin's Modified Melnikov Function

In further work, it is assumed that $\Omega \ll 1$. This means that z is a slow time variable. The modified Melnikov function, $M_w(t_0, z)$, which can detect the homoclinic intersections at a three-dimensional Poincaré section based at t_0 as given by Wiggins (1988) is

$$M_w(t_0, z) = \int_{-\infty}^{\infty} (f_1 g_2 - f_2 g_1)dt + \Omega\int_{-\infty}^{\infty} \{f_1(\partial f_2/\partial z) - f_2(\partial f_1/\partial z)\}t\,dt. \quad (36)$$

Here

$$f_1 = f_1\{q(t, z)\}, \quad f_2 = f_2\{q(t, z)\},$$

$$g_1 = g_1\{q(t, z), (t + t_0)\},$$

$$g_2 = g_2\{q(t, z), (t + t_0)\} \quad \text{and}$$

$$q(t, z) = \{\theta_1(t, z), \theta_2(t, z)\}.$$

Use of Eqs. (32), (33), and (34) in Eq. (36) and subsequent simplification leads to

$$M_w = -8\pi^2\epsilon_1\epsilon_2\{1 + (\epsilon_4/\epsilon_2)\sin(2\pi z)\}(1 - \cos\alpha) - 8\pi^2\epsilon_3 \sin(2\pi t_0)K_{uw} + 16\pi^2\epsilon_4\Omega\cos(2\pi z)K_{vw}. \quad (37)$$

The integrals K_{uw} and K_{vw} are given by

$$K_{uw} = \int_0^\alpha \cos[\mu(z)\ln\{\tan(0.25(\alpha - \phi))/\tan(0.25\alpha)\}] \times \cos(\alpha - \phi)d\phi \quad (38)$$

$$K_{vw} = \int_0^{\alpha} [4\pi^2\epsilon_2\{\mu(z)\}] \ln\{\tan(0.25(\alpha - \phi))/\tan(0.25\alpha)\}]$$
$$\times \sin(\alpha - \phi)d\phi \quad (39)$$

If $\epsilon_3 = 0$ in Eq. (37), the Melnikov function with only the vertical force acting is obtained.

Numerical Results

First the Melnikov function, M_{h1}, as given by Eq. (15) is considered. Given α, Ω, ϵ_4, and ϵ_1, the following relation between ϵ_2 and ϵ_3 is obtained for the zeros of M_{h1}

$$\epsilon_3 = (1/K_u \sin(2\pi t_0)\{\epsilon_4 \cos(2\pi\Omega t_0)K_v$$
$$- \epsilon_1\epsilon_2(1 - \cos\alpha)\}. \quad (40)$$

Now for any ϵ_2, one can find $\epsilon_3(t_0)$. The Melnikov boundary M_b in the parameter plane $\epsilon_3 - \epsilon_2$ is defined as the graph of inf $\{\epsilon_3(t_0)$ for all $t_0\}$ versus ϵ_2. In Figs. 5 and 6, the effect of changing the damping parameter ϵ_1 and the shape parameter α on M_b is shown. These results refer to the case when only the horizontal excitation is acting. In this case, the condition for initiation of rocking would be $\epsilon_3 = \epsilon_2 \tan\alpha$. These rocking initiation curves (RIC) are also shown in these figures. When the effect of vertical excitation is included homoclinic intersections of both the types are possible. For constructing the corresponding Melnikov boundaries one has to search for the zeros of Eqs. (15) and (26) such that $\epsilon_3(t_0)$ is minimized. However, it is not readily possible to fix up such a t_0 that minimizes ϵ_3 in the ϵ_3-ϵ_2 plane. It is therefore required to plot several graphs of $\epsilon_3(t_0)$ versus ϵ_2 for various t_0 values and then numerically find out the curve corresponding to inf $\{\epsilon_3(t_0)\}$ versus ϵ_2. This final result is shown in Fig. 7. In Fig. 8, to locate the modified boundary of Wiggins, Eq. (37) is set equal to zero and several graphs of inf $\{\epsilon_3(t_0, z)\}$ versus ϵ_2 are plotted for different values of z. The modified boundary M_w corresponds to $z = 0.5$ except for extremely small values of ϵ_2. This result is also compared in Fig. 8 with the usual Melnikov boundary M_b as given by Eq. (15). For the parameters chosen the modified boundary is weaker than the Melnikov boundary. However, Wiggin's modified approach will be valid for large values of ϵ_4 also.

Symmetric One-Period Orbit

The Melnikov boundary, for which numerical results are presented, divides the parameter space into two regions. Below this boundary only periodic solutions are possible. Above the boundary, however, the solutions are not necessarily periodic. With this in view it would be interesting to ask whether boundaries for specific types of periodic solutions can also be obtained. This requires extensive numerical work and is not undertaken here. On the other hand, an analytical solution is found for a symmetric one period orbit. No assumption on the slenderness of the block is made, but the damping is assumed to be small so that, the discontinuity at $\theta = 0$ can be smoothed. The solution valid in the interval $(-\alpha \le \theta \le \alpha)$ is taken in the form

$$\theta(t) = A\cos\beta,\ \dot{\theta}(t) = -2\pi A \sin\beta \quad (41)$$

where, $\beta = 2\pi t + \phi$. If A and ϕ are slowly varying, the classical averaging method leads to

$$\dot{A} = X_1 + X_2\cos\phi + X_3\sin\phi \quad (42)$$

$$\dot{\phi} = Y_1 + Y_2\cos\phi + Y_3\sin\phi \quad (43)$$

where

$$X_1 = -2\epsilon_1 A^2$$

$$X_2 = \pi\epsilon_3(J_0 + J_2)\cos\alpha$$
$$+ (8/3)\epsilon_3\{J_1 + (7/5)J_3 + (23/35)J_5\}\sin\alpha$$

$$X_3 = -(4/3)\epsilon_3 J_1 \sin\alpha$$

$$Y_1 = -\pi + 4(\epsilon_2/A)\{J_0 - (2/3)J_2 - (2/5)J_4\}\sin\alpha$$
$$- 2\pi(\epsilon_2/A)J_1\cos\alpha$$

$$Y_2 = (5/6)(\epsilon_3/A)J_1\sin\alpha$$

$$Y_3 = \pi(\epsilon_3/A)\{-J_0 + J_2\}\cos\alpha + (\epsilon_3/A)$$
$$\times\{(-25/6)J_1\sin\alpha + (16/15)J_3 + (17/210)J_5)\}. \quad (44)$$

Here $J_n(A)$ are Bessel functions of the first kind and order n. While deriving the above expressions the first three terms in the Fourier expansions,

$$\cos(A\cos\beta) = J_0(A) - 2J_2(A)\cos 2\beta$$
$$+ 2J_4(A)\cos 4\beta - \dots$$

$$\sin(A\cos\beta) = 2J_1(A)\cos\beta - 2J_3(A)\cos 3\beta$$
$$+ 2J_5(A)\cos 5\beta - \dots \quad (45)$$

have been retained. In the steady state, $\dot{A} = \dot{\phi} = 0$. This leads to the transcendental equations

$$(X_3Y_2 - X_2Y_3)^{-2}\{(X_2Y_1 - X_1Y_2)^2$$
$$+ (X_3Y_1 - X_1Y_3)^2\} - 1 = 0 \quad (46)$$

$$\tan\phi = (X_3Y_1 - X_1Y_3)^{-1}(X_1X_2 - X_2Y_1). \quad (47)$$

Solution of these equations leads to the steady-state values of A and ϕ.

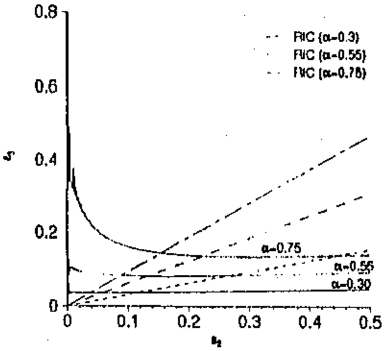

Fig. 6 Effect of shape parameter on M_b, only horizontal forcing ϵ_1 = 0.20

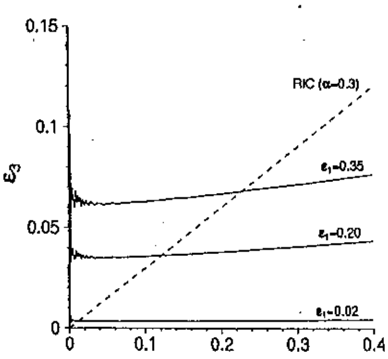

Fig. 5 Effect of damping on M_b, only horizontal forcing α = 0.3

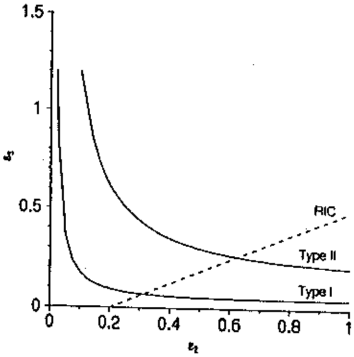

Fig. 7 Effect of vertical excitation on M_b $\epsilon_1 = 0.025$, $\alpha = 0.55$, $\epsilon_4 = 0.2$, $\Omega = 0.2$

An advantage of analytical approaches is that one can perform a stability analysis on the solution. It is easily seen that the divergence of the vector field $V_f = \{f_1(f_2 + g_2)\}^T$ as given by Eq. (4) is

$$(\nabla V_f) = -2\epsilon_1\theta_2 \operatorname{sgn}(\theta_2)\delta(\theta_1). \tag{48}$$

Since $\theta_2 \operatorname{sgn}(\theta_2) > 0$, the phase-space of the system contracts by jumps whenever $\theta_1 = 0$. But the assumed continuous one period orbit, given by Eq. (41) cannot account for these jumps in every cycle. However, on an average with the assumed solution the divergence over one cycle is

$$\langle(\nabla V_f)\rangle = -2\epsilon_1(1/2\pi)\int_0^{2\pi}(-2\pi A \sin\beta) \times \operatorname{sgn}(\sin\beta)\delta(\cos\beta)d\beta = -4A\epsilon_1. \tag{49}$$

Over a long period of time this average divergence contracts the phase space exponentially by the same amount as given by Eq. (48). Thus the stability of the averaged steady-state solution (A, ϕ) can be studied by considering the variational equation

$$\ddot{v} + 4A\epsilon_1\dot{v} + \{-4\pi^2\epsilon_2 \cos(\alpha \operatorname{sgn}\theta - \theta) + 4\pi^2\epsilon_3 \sin(2\pi t)\sin(\alpha \operatorname{sgn}\theta - \theta)\}v = 0 \tag{50}$$

Here θ is a known periodic function and hence through the eigenvalues of the Floquet transition matrix, one can study whether the variation v grows or decays exponentially. This in turn establishes the stability boundary of the one-period solution given by Eq. (41).

The transcendental Eqs. (46) and (47) have been solved iteratively to obtain several possible symmetric one period solutions, for a block under only horizontal excitation. In Fig. 9, one such solution is presented and compared with the exact solution numerically obtained using the Runge-Kutta scheme. It is observed that the comparison is very favourable. The stability boundary of these solutions in the parameter plane $(\epsilon_2 - \epsilon_3)$ is presented in Fig. 10 along with the Melnikov boundary for homoclinic intersections of Type I and the corresponding RIC.

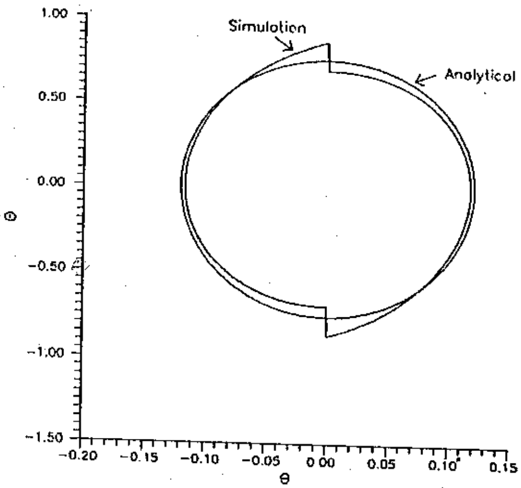

Fig. 9 Numerically and analytically computed symmetric 1—period orbits $\epsilon_1 = 0.2$, $\epsilon_2 = 0.03$, $\epsilon_3 = 0.11$, $\alpha = 0.55$

Discussion and Conclusions

The purpose of this paper has been to bring into focus the importance of the Melnikov function in understanding the dynamical behavior of a free-standing block rocking on a rigid base. The presence of homoclinic trajectories in the unperturbed phase space allows one to construct the Melnikov function and to check for the possibility of homoclinic intersections, when the system is perturbed by damping and external forcing. In previous studies, the assumption of piecewise linearity has been made to find the Melnikov boundary. Here this assumption has not been used. The effect of vertical excitation has also been included. There are two ways in which the vertical excitation can be handled. First, the amplitude can be taken to be small and thus the excitation can be treated as a small perturbation. The second approach is due to Wiggins wherein, the frequency parameter of the vertical excitation is treated as a small quantity. The Melnikov boundary (M_b) partitions the parameter space such that below the boundary only periodic trajectories are pos-

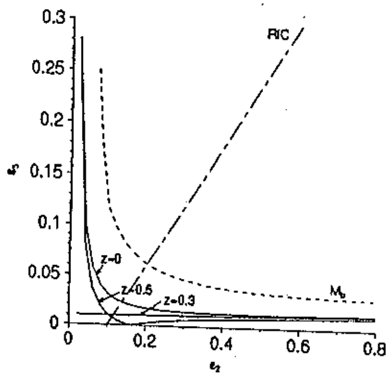

Fig. 8 Wiggin's modified boundary M_{bw} $\epsilon_1 = 0.025$, $\epsilon_4 = 0.1$, $\alpha = 0.55$, $\Omega = 0.1$

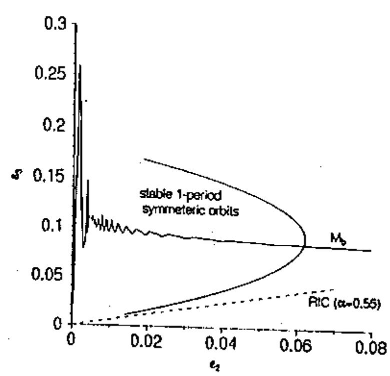

Fig. 10 Stability boundary of 1-period solutions and Melnikov boundary $\epsilon_1 = 0.2$, $\epsilon_4 = 0.0$, $\alpha = 0.55$

sible. In the present problem, an interesting question is the toppling of the block. For this to happen the trajectories have to cross the unperturbed separatrix. Hence, one can take the boundary M_h or the modified boundary M_{hm} as limiting curves below which toppling cannot occur. Thus crossing this boundary in the parameter plane would be a necessary condition for toppling of the block. It is interesting to note that a one period symmetric orbit not hitherto reported in the literature can be analytically obtained. The stability of this solution can also be analytically studied. This provides a further boundary in the (ϵ_2-ϵ_3) parameter plane (Fig. 10) to refine the region where complicated response may be possible. However, the present results provide only necessary conditions. Further structuring of this region to demarcate multiperiod, quasi-period and toppling solutions is necessary. The concept of lobe dynamics and the transport of the points in phase space across the pseudo-separatrix (Wiggins, 1992) may prove useful in studying the toppling characteristics of the block.

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APPENDIX

Proposition A1

Let $q_s(t - t_0)$ and ${}^{\epsilon}q_s(t, t_0)$ denote the unperturbed and perturbed stable homoclinic trajectories for the system of Eq. (4) on a Poincaré section based at t_0. Let ${}^{\epsilon}q_s(t_0, t_0) - q_s(0) = O(\epsilon)$. Then ${}^{\epsilon}q_s(t, t_0) - q_s(t - t_0) = O(\epsilon)\ \forall t \in (t_0, \infty)$.

A similar proposition holds true for the unstable homoclinic trajectory with time reversed.

Proof. Let $\mathbf{f}$ and $\mathbf{g}$ be vector-valued functions as given by Eq. (11). The function $\mathbf{g}$ may be equivalently represented as

$$\mathbf{g}({}^{\epsilon}q_s, t, \epsilon) = \epsilon\bar{\mathbf{g}}({}^{\epsilon}q_s, t, \epsilon). \quad \text{(A1)}$$

This leads to

$$|{}^{\epsilon}q_s(t, t_0) - q_s(t - t_0)| \le |{}^{\epsilon}q_s(t_0, t_0) - q_s(0)| + \int_{t_0}^{t} |\mathbf{f}({}^{\epsilon}q_s(\xi, t_0)) - \mathbf{f}(q_s(\xi - t_0))|\, d\xi + \epsilon \int_{t_0}^{t} |\bar{\mathbf{g}}({}^{\epsilon}q_s(\xi, t_0), \xi, t)|\, d\xi. \quad \text{(A2)}$$

Referring to Eq. (4), the function $\bar{\mathbf{g}}$ may be decomposed as

$$\bar{\mathbf{g}} = \bar{\mathbf{g}}^{c} + \bar{\mathbf{g}}^{d} \quad \text{(A3)}$$

where

$$\bar{\mathbf{g}}^{d} = \{0 - \theta_2|\theta_2|\delta(\theta_1)\}^{T}. \quad \text{(A4)}$$

Since $\mathbf{f}$ and $\bar{\mathbf{g}}^{c}$ are C^0 functions, hence $\exists M \ge 0$, and $L \ge 0$, such that in an interval $[t_0, t_1]$, where $t_1 > t_0$ and $t_1 - t_0 = O(1)$,

$$|\bar{\mathbf{g}}^{c}({}^{\epsilon}q_s, t, \epsilon)| \le M$$

and

$$|\mathbf{f}({}^{\epsilon}q_s(\xi)) - \mathbf{f}(q_s(\xi))| \le L|{}^{\epsilon}q_s(\xi) - q_s(\xi)|. \quad \text{(A5)}$$

Further, let there be I impacts of the block with the ground in $[t_0, t_1]$. It may then be readily shown that

$$\int_{t_0}^{t} |\bar{\mathbf{g}}^{d}({}^{\epsilon}q_s(\xi), \xi, \epsilon)|\, d\xi = K, \quad \text{(A6)}$$

where $K = \Sigma_{i=0}^{I} |\theta_{2i}|_{\theta_1 = 0}$.

In other words, K is the sum of the absolute velocities at impacts. Substitution of (A6) and (A7) in (A3) followed by the use of Gronwall's lemma leads to

$$|{}^{\epsilon}q_s - q_s| \le [|{}^{\epsilon}q_s(t_0, t_0) - q_s(0)| + \epsilon(K + M/L)] \exp[L(t - t_0)]. \quad \text{(A7)}$$

Hence $\exists$ a constant Q independent of ϵ, such that ${}^{\epsilon}q_s - q_s \equiv O(\epsilon)$ for $t_0 \le t \le t_0 + Q/L$. Since ${}^{\epsilon}q_s$ is a stable manifold, ${}^{\epsilon}q_s - q_s \equiv O(\epsilon)$ for $t_0 \le t < \infty$. □

Proposition A2

A transversal intersection of ${}^{\epsilon}q_s$ and ${}^{\epsilon}q_u$ is not possible at $\theta_1 = 0$.

Proof. ${}^{\epsilon}q_s$ and ${}^{\epsilon}q_u$ has a transversal intersection at a point p, iff

$$T_p({}^{\epsilon}q_s) + T_p({}^{\epsilon}q_u) = R^2. \quad \text{(A8)}$$

At $\theta_1 = 0$, ${}^{\epsilon}q_s$ and ${}^{\epsilon}q_u$ undergo a jump along the θ_2-axis. Therefore the only way ${}^{\epsilon}q_s$ and ${}^{\epsilon}q_u$ may intersect is along a line on the θ_2-axis. Let $p' \in {}^{\epsilon}q_s \cap {}^{\epsilon}q_u$ be one such intersection. Then

$$T_{p'}({}^{\epsilon}q_s) + T_{p'}({}^{\epsilon}q_u) = R^1. \quad \square \quad \text{(A9)}$$