Conditional Linearization in Nonlinear Random Vibration

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Non Linear : Random : Vibration

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Abstract: In this paper, an improved probabilistic linearization approach is developed to study the response of nonlinear single degree of freedom (SDOF) systems under narrow-band inputs. An integral equation for the probability density function (PDF) of the envelope is derived. This equation is solved using an iterative scheme. The technique is applied to study the hardening type Duffing's oscillator under narrow-band excitation. The results compare favorably with those obtained using numerical simulation. In particular, the bimodal nature of the PDF for the response envelope for certain parameter ranges is brought out.

INTRODUCTION

Random vibration problems of nonlinear systems have been tackled using a variety of approaches. The most popular and perhaps the simplest method is the equivalent linearization (Lin 1967) technique (ELT). Alternatives include perturbation (Crandall 1973), Fokker-Planck equation (Spanos 1982), closure (Ibrahim 1978; Iyengar and Dash 1978), and higher-order linearization (Iyengar 1988a) methods. Except for simple systems under broad-band excitation, exact solutions are not available. A problem of considerable practical interest is a nonlinear system under narrow-band excitation. For example, this can be the model of power-plant equipment vibrating under a floor excitation that is narrow-banded. The floor excitation itself may be caused by a ground-level earthquake that gets strongly filtered by the primary structure to induce narrow-band inputs to secondary systems.

Approximate analysis of the Duffing oscillator under narrow-band excitation has been presented by several investigators. Lyon et al. (1961), through experiments, and Lennox and Kuak (1976) and Iyengar (1988b), through numerical simulations, showed that under certain conditions the amplitudes show transitions between two levels. Efforts to understand this through the usual ELT have not been very successful. ELT leads to moment equations (Davies and Nandlall 1981; Iyengar 1989) showing multiple values for the response variance, which in fact must have a unique value. The fact that the response process may stay at two different amplitude levels could only mean that the underlying process is highly non-Gaussian and that the probability density function (PDF) of the peaks of this process may be bimodal. The success of approximate methods in deterministic systems is due to the fact that the forms of the assumed solution have been reasonably correct. In random vibration problems, if one were to correctly estimate the probabilistic structure of the response process, it is necessary for the approach to have inbuilt flexibility to reflect possible non-Gaussianness. Thus a solution form such as $x = r\cos(\lambda t - \theta)$ for the Duffing oscillator under harmonic excitation leads to a cubic polynomial for r with three possible solutions. If this property has to get probabilistically reflected in the narrow-band excitation problem, one must arrive at a functional equation for the PDF $p(r)$, the solution of which may show multiple extrema. This line of argument leads to a new kind of linearization principle in which the nonlinear system is approximated by a linear system with random coefficients. A simpler version of this approach was described by Iyengar (1992) to obtain the response variance of the Duffing oscillator under narrow-band excitation. The present paper refines and extends the aforementioned to show how a functional equation can be derived for the PDF of the response peaks. The solution of this equation shows that the peak PDF can be multimodal for some parameter values. Comparison with simulated results is found to be favorable.

CUBIC OSCILLATOR

The nonlinear system considered is the hardening Duffing's oscillator excited by a narrow-band noise. The system is described by

$$\ddot{y} + 2\eta\omega\dot{y} + \omega^2 y + \alpha\omega^2 y^3 = f(t) \quad (1)$$

Here the excitation $f(t)$ is a filtered white noise given by

$$\ddot{f} + 2\xi\lambda\dot{f} + \lambda^2 f = W(t) \quad (2)$$

$$\langle W(t_1)W(t_2)\rangle = I\delta(t_2 - t_1); \quad \sigma_f^2 = I/(4\xi\lambda^3) \quad (3a,b)$$

The input bandwidth and center frequency are controlled by ξ and λ, respectively. For the linear case, with $\alpha = 0$, the steady-state variance is

$$\sigma_l^2 = \frac{(\sigma_f^2/\omega^4)[1 - \bar{\lambda}^2 + (1 + \bar{\xi}\bar{\lambda})(\bar{\lambda}^2 + 4\bar{\xi}\eta\bar{\lambda})]}{[(1 - \bar{\lambda}^2)^2 + 4\eta\bar{\lambda}(1 + \bar{\xi}\bar{\lambda})(\bar{\xi} + \eta\bar{\lambda})]} \quad (4a)$$

$$\bar{\lambda} = \lambda/\omega; \quad \bar{\xi} = \xi/\eta \quad (4b,c)$$

Under the influence of white-noise excitation, the hardening system shows the response variance to be below its linear value, namely σ_l^2. However, when f is narrow-banded, the variance for a range of values of $\bar{\lambda}$ can be greater than σ_l^2. With this in view, it is convenient to introduce the nondimensional variables

$$x = y/\sigma_l; \quad \alpha\sigma_l^2 = \varepsilon < 1 \quad (5a,b)$$

Now, the nonlinear system to be analyzed will be

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x + \varepsilon\omega^2 x^3 = f(t)/\sigma_l \quad (6)$$

CONDITIONAL LINEARIZATION

For small values of ε and ξ, the response is expected to be at the frequency λ. Therefore an approximation to the response will be

$$x(t) = x_m \sin(\lambda t - \theta) \quad (7)$$

Here x_m is the random amplitude that is yet unknown. The nonlinear term x^3 is replaced by γx and the mean square error over one cycle of oscillation is minimized as in the harmonic linearization technique (Spanos and Iwan 1979). Since the amplitude and phase are slowly varying over one cycle, x_m and θ can be treated as random variables. This gives

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$$\gamma = 0.75x_m^2 \tag{8}$$

The resulting linear equation is

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2(1 + 0.75\varepsilon x_m^2)x = f/\sigma_f \tag{9}$$

Here x_m is the unknown random peak amplitude, and thus strictly speaking, this equation is still nonlinear. However for random vibration analysis, since x_m is treated as a random variable, the system is conditionally linear. In other words, the process $(x|x_m)$ is Gaussian. The conditional variance of x and $\dot{x}$ are easily found to be

$$\mu_1^2 = \langle x^2|x_m\rangle \tag{10a}$$

$$\mu_1^2 = \frac{\beta^2[D + (1 + \xi\bar{\lambda})(\bar{\lambda}^2 + 4\xi\eta\bar{\lambda})]}{(1 + 0.75\varepsilon x_m^2)[D^2 + 4\eta\bar{\lambda}(1 + \xi\bar{\lambda})\{\xi(1 + 0.75\varepsilon x_m^2) + \eta\bar{\lambda}\}]} \tag{10b}$$

$$D = (1 + 0.75\varepsilon x_m^2 - \bar{\lambda}^2) \tag{10c}$$

$$\beta = \sigma_f/(\omega^2\sigma_f) \tag{10d}$$

$$\mu_2^2 = \langle \dot{x}^2|x_m\rangle/\bar{\lambda}^2 \tag{11a}$$

$$\mu_2^2 = \frac{\beta^2(1 + \xi\bar{\lambda})}{[D^2 + 4\eta\bar{\lambda}(1 + \xi\bar{\lambda})\{\xi(1 + 0.75\varepsilon x_m^2 + \eta\bar{\lambda}\}]} \tag{11b}$$

The variance $\langle x^2\rangle$ of x can be found if $p(x_m)$ is known. Since x_m is the maximum of x, there will be a compatibility condition to be satisfied. Previously, under the assumption that x_m is Rayleigh distributed, Iyengar (1992) found out the response variance by solving the moment compatibility condition

$$\sigma_x^2 = \int_0^\infty \langle x^2|x_m\rangle p(x_m)\,dx_m \tag{12}$$

where

$$p(x_m) = (x_m/\sigma_x^2)\exp(-0.5x_m^2/\sigma_x^2) \tag{13}$$

The Rayleigh assumption for x_m excludes possible multimodality in the PDF and thus is not the right choice. Also, it is not clear what could be the proper functional form for the PDF. To circumvent this difficulty, it is argued as follows. An approximation of the form $x = x_m \sin(\lambda t - \theta)$ is strictly valid only for a linear system ($\varepsilon = 0$) under sinusoidal excitation. Thus in the harmonic linearization using (7), x_m has to include the effect of nonlinearity if one seeks an improvement beyond (12) and (13).

PERTURBATION SERIES FOR x_m

Here the effect of nonlinearity is included through a perturbation power series in ε. The solution of (6) under a sinusoidal excitation may be taken as.

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \tag{14}$$

where the zero order solution is

$$x_0 = r\sin(\lambda t - \theta) \tag{15}$$

With this, it easily follows that

$$\ddot{x}_1 + 2\eta\omega\dot{x}_1 + \omega^2 x_1 = -\omega^2 x_0^3 \tag{16}$$

$$\ddot{x}_2 + 2\eta\omega\dot{x}_2 + \omega^2 x_2 = -3\omega^2 x_0^2 x_1 \tag{17}$$

Solutions for x_1 and x_2 can be obtained in terms of r and θ. For example

$$x_1(t) = -(0.75r^3/D_1)\sin(\lambda t - \theta - \delta_1) + (r^3/4D_2)\sin(3\lambda t - 3\theta - \delta_2) \tag{18}$$

$$x_2(t) = (27r^5/16D_1^2)\sin(\lambda t - \theta - 2\delta_1) + (3r^5/16D_1D_2)\sin(\lambda t - \theta - 2\delta_2) - (9r^5/16D_1D_2)\sin(3\lambda t - 3\theta - \delta_1 - \delta_2) - (3r^5/8D_2^2)\sin(3\lambda t - 3\theta - 2\delta_2) + (3r^5/16D_2D_3)\sin(5\lambda t - 5\theta - \delta_2 - \theta_3) \tag{19}$$

where

$$D_1 = 1/\sqrt{[(1 - \bar{\lambda}^2)^2 + (2\eta\bar{\lambda})^2]} \tag{20a}$$

$$D_2 = 1/\sqrt{[(1 - 9\bar{\lambda}^2)^2 + (6\eta\bar{\lambda})^2]} \tag{20b}$$

$$D_3 = 1/\sqrt{[(1 - 25\bar{\lambda}^2)^2 + (10\eta\bar{\lambda})^2]} \tag{20c}$$

$$\delta_1 = \tan^{-1}[2\eta\bar{\lambda}/(1 - \bar{\lambda}^2)] \tag{20d}$$

$$\delta_2 = \tan^{-1}[6\eta\bar{\lambda}/(1 - 9\bar{\lambda}^2)] \tag{20e}$$

$$\delta_3 = \tan^{-1}[10\eta\bar{\lambda}/(1 - 25\bar{\lambda}^2)] \tag{20f}$$

Now, collecting and decomposing the higher-frequency components in terms of $\sin(\lambda t - \theta)$ and $\cos(\lambda t - \theta)$

$$x(t) = [B_1 + K_1\cos(2\lambda t - 2\theta - \delta_2) + K_2\cos(2\lambda t - 2\theta - \delta_1 - \delta_2) + K_3\cos(2\lambda t - 2\theta - 2\delta_2) + K_4\cos(4\lambda t - 4\theta - \delta_2 - \delta_3)]\sin(\lambda t - \theta) + [B_2 + K_1\sin(2\lambda t - 2\theta - \delta_2) + K_2\sin(2\lambda t - 2\theta - \delta_1 - \delta_2) + K_4\sin(2\lambda t - 2\theta - 2\delta_2) + K_4\sin(4\lambda t - 4\theta - \delta_2 - \delta_3)]\cos(\lambda t - \theta) \tag{21}$$

where B_1, B_2, K_1, K_2, K_3, and K_4 = coefficients readily obtained by substituting (18) and (19) in the perturbation series (14). This can be further represented in the form of (7). Thus

$$x_m^2 = r^2 - \varepsilon C_1 r^4 + \varepsilon^2 C_2 r^6 + \cdots \tag{22}$$

where $C_1 = (3/2D_1)\cos\delta_1$; and $C_2 = [(9/16D_1^2) + (27/8D_1^2)\cos 2\delta_1 + (3/8D_1D_2)\cos(\delta_1 + \delta_2) + (1/16D_2^2)]$.

INTEGRAL EQUATION FOR $p(r)$

Since x_m is expressed in terms of r, one can treat r as the unknown. The conditioned process $(x|r)$ is Gaussian, the moments of which can be obtained from (10) and (11) where x_m^2 is given by (22). Now, for a sample of the narrow-banded response process $x(t)$, x_m may be treated as $\sup[x(t)|t \in (0, 2\pi/\lambda)] = \sqrt{x^2 + \dot{x}^2/\lambda^2}$. Thus focusing attention on the conditioned Gaussian process $(x|r)$, the steady-state PDF of the conditioned maxima $(x_m|r)$ can be found from (9) as

$$p(x_m|r) = (x_m/\mu_1\mu_2)\exp[-x_m^2(\mu_1^2 + \mu_2^2)/4\mu_1^2\mu_2^2]I_0[x_m^2(\mu_1^2 - \mu_2^2)/4\mu_1^2\mu_2^2] \tag{23}$$

where μ_1 and μ_2 = conditional variances as given by (10) and (11); and I_0 = modified Bessel's function of order zero. Hence, two possible representations for x_m have been argued out. Although the first one, as given by (22), is an algebraic relationship between x_m and r, the second one, given by (23), is simply the envelope distribution of the conditionally Gaussian process $(x|r)$. These two relations should, in turn, be compatible. The compatibility condition is

$$\int_0^\infty p(x_m|r)p(r)\,dr = p(x_m) \tag{24}$$

Here the right-hand side is expressible through (22) in terms of r. Hence (24) is an integral equation for finding $p(r)$. This is solved through an iterative scheme to arrive at $p(r)$ and hence $p(x_m)$.

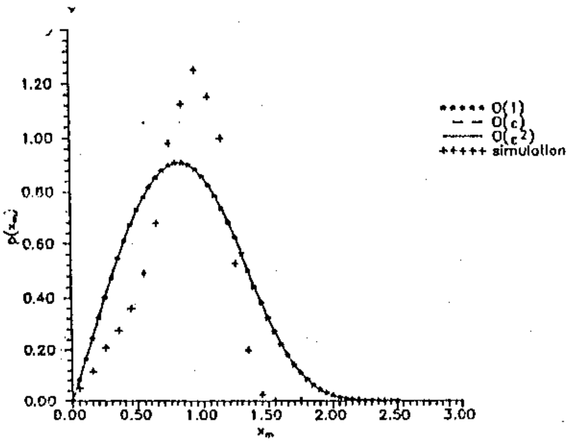

FIG. 1. Probability Density Function for Maximum $\bar{\lambda} = 1$

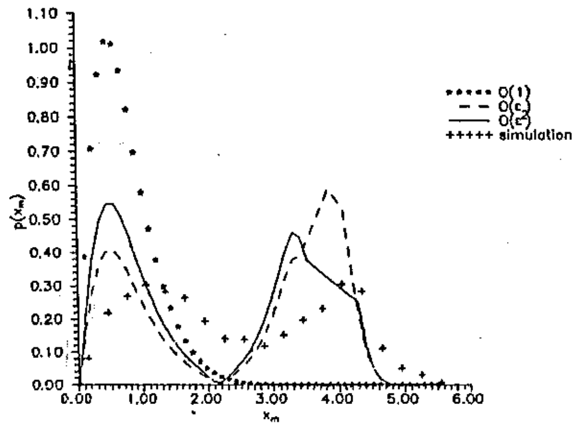

FIG. 2. Probability Density Function for Maximum $\bar{\lambda} = 2$

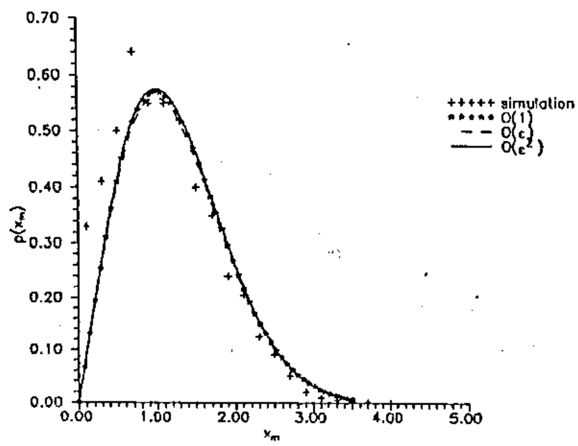

FIG. 3. Probability Density Function for Maximum $\bar{\lambda} = 3$

ITERATIVE SCHEME

To start with, r is assumed to be Rayleigh distributed, i.e.

$$p_0(r) = (r/\sigma^2)\exp(-r^2/2\sigma^2) \quad (25)$$

This is the zero iterate for $p(r)$, the unknown in (24). However, in this equation σ^2 is an unknown; but this can be obtained by solving

$$\int_0^\infty \int_0^\infty p(x_m|r)p_0(r)\,dr\,dx_m = 1 \quad (26)$$

where $p_0(r)$ is substituted in (24) to obtain the corresponding $p_0(x_m)$. Now, since (22) is a memory-less transformation connecting r and x_m, the previous $p_0(x_m)$ is transformed to obtain the next iterate $p_1(r)$. This process is repeated until $p_i(r)$ and $p_i(x_m)$ converge for all values of r.

NUMERICAL RESULTS

Eqs. (25) and (26) have been solved for a nonlinear system with $\varepsilon = 0.3$, $\eta = 0.08$, and $\xi = 0.02$. The frequency parameter has been varied with values $\bar{\lambda} = 1$, 2, and 3. The numerically simulated variance and its comparison with the equivalent linear predictions have been previously reported by Iyengar (1992) and thus are not repeated here. In Figs. 1, 2, and 3, the present theoretical predictions for $p(x_m)$ are compared with numerical simulations. The iterative solution to (24) converged in about 10 iterations in all the cases.

In all the figures, the ε^0, ε^1, and ε^2 order approximations have been reported. For $\bar{\lambda} = 1$ and 3, the difference among the three approximations are negligible. In the case of $\bar{\lambda} = 2$, the ε^0 order approximation is inadequate to predict the biomodal peak distributions. It is clear that, in this case, inclusion of higher-order terms in (22) may lead to still better results. Keeping in view the fact that the simulated results refer to the peak PDF whereas the prediction is for the envelope PDF, the comparison between the two results is satisfactory.

SUMMARY AND CONCLUSIONS

The present approach is essentially an extension of and an improvement over the probabilistic linearization scheme earlier introduced by Iyengar (1992). In contrast with the earlier work, the present technique directly obtains the PDF for the envelope by deriving an integral equation that needs to be solved iteratively. To carry out the iterative scheme, it is important to establish a functional relationship between r and x_m, the enveloping processes of the zero order and higher orders, respectively, and aproximated over one cycle of the forcing as random variables due to their slowly varying nature. This, in turn, is accomplished by expanding the solution process in terms of a power series in ε, the nonlinearity parameter. The results show a considerable improvement over the earlier approach, especially for the bimodal case corresponding to $\bar{\lambda} =$ 2. The approach, however, may not be accurate enough for higher values of ε. It would also be of interest to see how the present technique works for other classes of nonlinear oscillators, for example, the limit cycle and the hysteretic systems.

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APPENDIX II. NOTATION

The following symbols are used in this paper:

B_1, B_2, K_1, K_2, K_3 = coefficients in representation for x;
C_1, C_2 = coefficients in functional relationship between x_m and r;
D, β = parameters in expressions for conditional variances;
D_1, D_2, D_3 = frequency responses;
$f(t)$ = input random process obtained as output of linear filter;
I = strength of white noise;
I_0 = modified Bessel's function of order 0;
i = index;
$p(x_m)$ = PDF for x_m;
$p(r)$ = PDF for r;
$p(x_m|r)$ = conditional PDF for x_m given r;
$p_i(x_m)$ = iterates for $p(x_m)$;
$p_i(r)$ = iterates for $p(r)$;
r, x_m = random amplitudes;
t, t_1, t_2 = time;
$W(t)$ = Gaussian white noise;
$x(t)$ = nondimensionalized random process corresponding to $y(t)$;
$x_0(t), x_1(t), x_2(t)$ = terms of various orders in perturbed series solution for $x(t)$;
$y(t)$ = random process representing solution for Duffing's equation under $f(t)$;
α, ε = nonlinearity parameters;
γ = coefficient in equivalent linear term;
δ = Dirac delta function;
η, ξ = damping ratios;
$\bar{\eta}, \bar{\xi}$ = normalized damping ratios;
$\theta, \delta_1, \delta_2, \delta_3$ = phase lags in radians;
λ, ω = natural frequencies;
$\bar{\lambda}$ = normalized natural frequency;
μ_1^2, μ_2^2 = conditional variances of x given x_m;
σ_f^2 = steady-state variance of process $f(t)$;
σ_l^2 = steady-state variance of linear equation corresponding to $\varepsilon = 0$; and
σ_x^2 = variance of process $x(t)$.