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ANALYSIS OF A SHALLOW ELLIPTIC PARABOLOID SHELL WITH EDGE BEAMS

by

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Analysis of a shallow elliptic paraboloid shell with edge beams

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The paper presents a method for the analysis of shallow elliptic paraboloid shells having edge beams. The bending theory equations are solved by Fourier methods. A shell has been analysed and the stress resultants are presented in the form of contours. This method of solution is formally exact; however, heavy computations are necessary for good accuracy.

Theoretical and experimental investigations on shells have clearly revealed that the stress distributions obtained by the use of the membrane theory are far from the actual state of stress. Especially for shallow shells without axial symmetry the assumption of momentlessness, on which the membrane theory is based, is highly misleading[3]. Furthermore, the membrane theory fails to show up the differences in the distribution of stresses that must occur when the boundary conditions are changed. Only a bending theory can yield a better picture of stresses in such shells.

Even while using a bending theory to analyse shells it has been usual to assume certain boundary conditions, known as the 'Navier conditions', to reduce the mathematical complexities. These conditions can never be satisfied physically and are far from the usual conditions occurring in practice.

Shells are usually supported on, or built into walls, diaphragms, or beams supported on columns. If a general method of solution can be obtained for the analysis of a shell supported on edge beams, the boundary conditions encountered in practice become particular cases of the general method. For example, by taking the flexural rigidity of an edge beam in the vertical plane to be infinite, that in the horizontal plane and also the torsional rigidity to be zero, a 'simple support' can be simulated. The case of a 'built-in support' is covered by taking the torsional and flexural rigidities of an edge beam to be infinite.

In this paper a shallow elliptic paraboloid shell has been analysed using the bending theory for shallow shells due to Marguerre[1]. The shell is assumed to be supported on edge beams which are in turn simply supported at their ends (*Fig* 1).

Basic equations

The equation of an elliptic paraboloid, with reference to the co-ordinates shown in *Fig* 1, is given by

$$z = \tfrac{1}{2}\,(rx^2 + ty^2) \qquad \ldots\ldots (1)$$

where r and t are constants.

Marguerre's equations for a shell of constant thickness h, having a middle surface defined by equation (1) and subjected to a uniformly distributed load q_z, take the form:

$$\nabla^4 F + Eh\,(tw_{xx} + rw_{yy}) = 0 \qquad \ldots\ldots (2a)$$

and

$$D\nabla^4 w - (rF_{yy} + tF_{xx}) = q_z \qquad \ldots\ldots (2b)$$

where w is the deflection of the shell, F is an Airy stress function and the biharmonic operator ∇^4 is given by

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\,\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

Suffixes denote partial differentiation.

The pair of equations (2) can be combined to give a single eighth order differential equation in terms of another function Φ,

$$D\nabla^8\Phi + Eh\Box^4\Phi - q_z = 0 \qquad \ldots\ldots (3)$$

where

$$\nabla^8 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^4$$

$$\Box^4 = \left(t\frac{\partial^2}{\partial x^2} + r\,\frac{\partial^2}{\partial y^2}\right)^2$$

$$w = \nabla^4\Phi \qquad \ldots\ldots (4a)$$

and

$$F = -Eh\,\Box^2\Phi \qquad \ldots\ldots (4b)$$

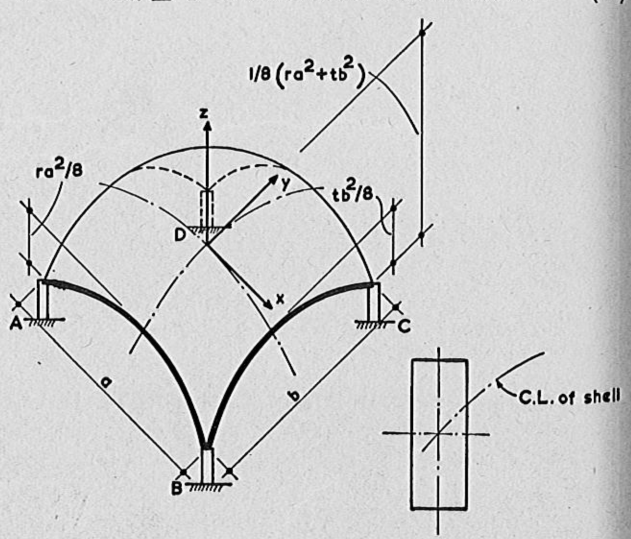

Fig 1 Shell supported on edge beams

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NOTATION

x, y, z	Cartesian co-ordinates	ν	Poisson's ratio
∇^4	Biharmonic operator	D	equals $Eh^3/12\,(1-\nu^2)$
F	Airy stress function	G	Modulus of rigidity
u, v, w	Deformations in x, y, and z directions, respectively.	$C, \bar{C}$	Torsional rigidity of the edge beams (torque per unit twist per unit length)
q_z	Uniformly distributed load intensity (positive in the z direction)		
Φ	Generalized stress-strain function	$I_y, \bar{I}_y, I_z, \bar{I}_z$	Second moments of area of sections of edge beams about y and z axis, respectively.
$\Box^2$	Pücher's operator		
E	Modulus of elasticity	$A_o, \bar{A}_o$	Sectional area of the edge beams
h	Shell thickness	Φ_o	Particular integral in Φ
a, b	Dimensions of the covered area	α	equals $n\pi/a$ where n is an odd integer
N_x, N_y, N_{xy}	In-plane stress resultants		
M_x, M_y, M_{xy}	Moment resultants	β	equals $m\pi/b$ where m is an odd integer.
Q_x, Q_y	Normal shears		

The stress resultants N_x, N_y, Q_x, Q_y, and N_{xy}, the moments M_x, M_y, and M_{xy}, and the displacement w can be obtained from the following relations :

$$\left.\begin{aligned} N_x &= F_{yy} \\ N_y &= F_{xx} \\ N_{xy} &= -F_{xy} \\ w &= \nabla^4\Phi \\ M_x &= -D\,(w_{xx} + \nu w_{yy}) \\ M_y &= -D\,(w_{yy} + \nu w_{xx}) \\ M_{xy} &= -D\,(1-\nu)\,w_{xy} \\ Q_x &= -D\,(w_{xxx} + w_{xyy}) \\ Q_y &= -D\,(w_{yyy} + w_{yxx}) \end{aligned}\right\} \quad \ldots\ldots\ldots(5)$$

The positive directions of resultants are shown in *Fig* 2.

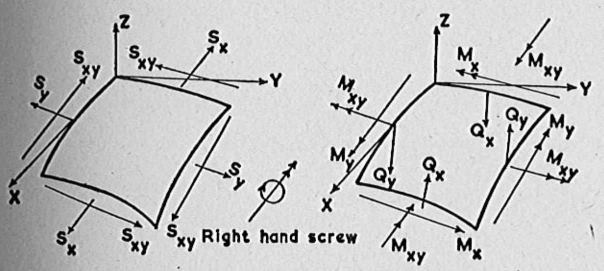

Fig 2 Positive forces and moments acting on an element of a shell

Boundary conditions

It is assumed that the edge beams and the shell are connected at the centre of the beam section. Since the shells are shallow the curvature of the edge beams is small, and it is assumed that the equations for straight beams are good enough for the purpose of forming the boundary conditions. These conditions are explicitly stated below for a typical edge at $y = b/2$[2].

For vertical equilibrium of an element of the beam

$$EI_y\, w_{xxxx} = z_y\, F_{xx} - z_x\, F_{xy} - D[w_{yyy} + (2-\nu)\, w_{xxy}] \quad \ldots\ldots\ldots\ldots\ldots\ldots(6a)$$

For horizontal equilibrium :

$$EI_z \left[\frac{1}{Gh}(F_{xy})_{xxx} + \frac{1}{Eh}(F_{yy} - \nu F_{xx})_{xxy} + (z_y\, w_x + z_x\, w_y)_{xxx} - (z_x w_x)_{xxy} \right] = F_{xx} + z_y\, D\,[w_{yyy} + (2-\nu)\, w_{xxy}] \quad \ldots\ldots(6b)$$

From consideration of the torsional equilibrium of an element,

$$C\, w_{yxx} + D\,(w_{yy} + \nu\, w_{xx}) = 0 \quad \ldots\ldots\ldots\ldots(6c)$$

and for strain compatibility in the longitudinal direction,

$$EA_o\, U_x = \int_{a/2}^{x} N_{yx}\, dx$$

which can be simplified to

$$\frac{A_o}{h}(F_{yy} - \nu F_{xx}) - EA_o\, z_x\, w_x - F_y \text{ (at } x = a/2) + F_y = 0 \quad \ldots\ldots\ldots\ldots\ldots(6d)$$

In addition to these, the solution must satisfy the condition

$$w = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6e)$$

at the supports $(\pm a/2, \pm a/2)$.

The boundary conditions at the other edges at $x = \pm a/2$ can be written down from similarity considerations.

General solution of the partial differential equation

Fourier methods are adopted throughout for the solution of the boundary value problem.

The solution of the basic equation (3) can be expressed as

$$\Phi = \Phi_o + \sum_{m=1,3,5..} X_m \cos \beta y + \sum_{n=1,3,5..} Y_n \cos \alpha x \quad \ldots.(7)$$

where Φ_o is the particular integral, X_m is a function of x, Y_n a function of y, $\alpha = n\pi/a$ and $\beta = m\pi/b$. The integers m and n take odd values only.

Taking $\Phi_o = \sum_{n=1,3,5..} k_n \cos \alpha x$, substituting in equation (3), and expressing the load intensity q_z as

$$q_z = \frac{q_z}{D} \sum f_n \cos \alpha x \quad \dots\dots (8)$$

we get,

$$k_n = \frac{q_z}{D} \frac{f_n}{\alpha^4 (\alpha^4 + Ehr^2/D)} \quad \dots\dots (9)$$

Substituting the second term of the right-hand side of equation (7) in equation (3), an ordinary linear differential equation in X_m is obtained. This yields the solution

$$X_{im} = \cosh \Upsilon_{im} x$$

where γ_{im} are the roots of the characteristic equation

$$(\Upsilon_m^2 - \beta^2)^4 + \frac{Eh}{D}(r\Upsilon_m^2 - t\beta^2) = 0 \quad \dots\dots (10)$$

The eight complex roots are explicitly given by

$$\Upsilon_{m\,(1,3,5,7)} = \pm\sqrt{\tfrac{1}{2}}\sqrt{[c_1 \pm \sqrt{(c_1^2 - 4c_2)}]} \dots (11a)$$

$$\Upsilon_{m\,(2,4,6,8)} = \pm\sqrt{\tfrac{1}{2}}\sqrt{[c_3 \pm \sqrt{(c_3^2 - 4c_4)}]} \dots (11b)$$

where

$$C_1 = 2\beta^2 + jr\sqrt{\frac{Eh}{D}},\ j = \sqrt{-1}$$

$$C_2 = \beta^4 + jt\,\beta^2\sqrt{\frac{Eh}{D}}$$

C_3 and C_4 are the conjugates of C_1 and C_2 respectively.

However, since $\cosh \Upsilon_m x$ is an even function, there are only four distinct forms of X_m and hence, in general,

$$X_m = \sum_{i=1,2,3,4} A_{im} \cosh \Upsilon_{im} x$$

where A_{im} are complex constants, to be determined from the boundary conditions and the restriction that X_m is real.

Similarly,

$$Y_n = \sum_{i=1,2,3,4} B_{in} \cosh \lambda_{in} y$$

where B_{in} are complex constants and λ_{in} are the four positive complex roots of the characteristic equation

$$(\lambda_n^2 - \alpha^2)^4 + \frac{Eh}{D}(t\lambda_n^2 - r\alpha^2)^2 = 0 \quad \dots\dots (12)$$

Thus Φ can be written as

$$\Phi = \sum_{n=1,3,5} k_n \cos \alpha x + \sum_{i=1,2,3,4} \left[\sum_{m=1,3,..} A_{im} \cosh \Upsilon_{im} x \cos \beta y + \sum_{n=1,3,..} B_{in} \cosh \lambda_{in} y \cos \alpha x \right] \dots (13)$$

Since the complex roots Υ_{2m} and Υ_{4m} are the conjugates of Υ_{1m} and Υ_{3m}, respectively, and Φ is a real function, it follows that A_{2m} and A_{4m} are the conjugates of A_{1m} and A_{3m}, respectively.

Substituting for Φ in equation (4a) the value obtained from equation (13), the displacement w can be expressed as

$$w = \sum_{n=1,3,...} k_n \alpha^4 \cos \alpha x + \sum_{i=1,2,3,4} \left[\sum_{m=1,3,..} (\Upsilon_{im}^2 - \beta^2)^2 A_{im} \cosh \Upsilon_{im} x \cos \beta y + \sum_{n=1,3,...} (\lambda_{in}^2 - \alpha^2)^2 B_{in} \cosh \lambda_{in} y \cos \alpha x \right] \quad ..(14a)$$

Substituting for Φ in equation (4b) the stress function F can be written as

$$F = \sum_{n=1,3,...} Eh\,\alpha^2 t\,k_n \cos \alpha x + Eh \sum_{i=1,2,3,4} \left[\sum_{m=1,3,..} (r\beta^2 - t\Upsilon_{im}^2) A_{im} \cosh \Upsilon_{im} x \cos \beta y + \sum_{n=1,3,...} (t\alpha^2 - r\lambda_{in}^2) B_{in} \cosh \lambda_{in} y \cos \alpha x) \right] \quad ..(14b)$$

Solution of the boundary value problem

All the boundary conditions (6a–d) are given in terms of w and F which are given by equations 14(a) and 14(b), respectively. By substituting these expressions in the boundary conditions a set of boundary equations in terms of A_{im} and B_{in} are obtained. For example, equation (6c) at the edge $y = b/2$ yields

$$\sum_i \left[\sum_n \left[D(\lambda_{in}^2 - \alpha^2)^2 (\lambda_{in}^2 - \nu\alpha^2) B_{in} \cosh \lambda_{in} \frac{b}{2} \cos \alpha x - C(\lambda_{in}^2 - \alpha^2)^2 \alpha^2 \lambda_{in} B_{in} \sinh \lambda_{in} \frac{b}{2} \cos \alpha x \right] - \sum_m C(\Upsilon_{im}^2 - \beta^2)^2 \beta\,\Upsilon_{im}^2 A_{im} \cosh \Upsilon_{im} x \right]$$

$$- \sum_n k_n D\nu\alpha^6 \cos \alpha x = 0 \quad \dots\dots (15)$$

Since this equation must hold good for any value of x, it is necessary to express all the terms of the equation in the form of a Fourier Series in x, so that this functional equation can be reduced to an equation in terms of the coefficients A_{im} and B_{in} only.

To achieve this, it would be necessary to express $\cosh \Upsilon_{im} x$ as a Fourier Series. This can be done with the help of Fourier Transforms.

For example,

$$\cosh \Upsilon_{im} x = \sum_{l=1,3,5,..}^{\infty} \frac{4l\pi \sin \frac{l\pi}{2}}{l^2\pi^2 + \Upsilon_{im}^2 a^2} \cosh \frac{\Upsilon_{im} a}{2} \cos \frac{l\pi x}{a}$$

Substituting this in equation (15), the equation reduces to

$$\sum_{n=1,2,3,4} \Bigg[-\sum_{m=1,3,..} A_{im}\, C\, (\gamma_{im}^2 - \beta^2)^2\, \beta\, \lambda_{im}^2 \sin\frac{m\pi}{2}$$

$$\times \frac{4n\pi \sin\frac{n\pi}{2}}{n^2\pi^2 + \lambda_{im}^2\, a^2} \cosh\frac{\gamma_{im} a}{2} + B_{in}\, D\, (\lambda_{in}^2 - \alpha^2)^2 \, (\lambda^2_{in} - \nu\alpha^2)$$

$$\times \cosh\frac{\lambda_{in}\, b}{2} - B_{in}\, C\, (\lambda_{in}^2 - \alpha^2)^2\, \alpha^2\, \lambda_{in}^2 \sinh\frac{\lambda_{in} b}{2} \Bigg]$$

$$- k_n D \nu \alpha^6 = 0$$

This equation appears in non-dimensional form in *Appendix I* as equation (*A*3).

Appendix I gives non-dimensional boundary equations and *Appendix II* gives the various Fourier Transforms required for their derivation.

The eight sets of equations (*Appendix I* equations *A*1–*A*4 and their counterparts for the edge $x = a/2$) can be solved to get the coefficients A_{im} and B_{in} by taking a finite number of terms of series, that is, by terminating the series in m and n after a few terms. For example, m and n can be restricted to have only four values, namely, 1, 3, 5 and 7. This truncation introduces approximations in the effective representations of functions by Fourier Series. Errors in the stresses and moments due to this truncation can be estimated by taking extra terms in m and n (say 1, 3, 5, 7 and 9) and comparing the two sets of results. However, such a procedure involves very tedious computations and it may not be always possible to make an error estimate.

After solving for A_{im} and B_{in}, the function Φ is completely known and hence the values of all stresses and moments can be found using equations (5). It is to be noted that A_{im} and B_{in} are complex but the function Φ and the stresses and moments will be real. Each of these will be the sum of a pair of series, one being the conjugate of the other.

Illustrative example

The dimensions of the shell and edge beam system analysed are given below:

$a, b = 20$ ft

$h/a = 0{\cdot}025$

$ar = 0{\cdot}2$

$at = 0{\cdot}25$

Depth of beam $= 3a/40$

Width of beam $= a/40$

All beams are identical.

Computations up to the stage of formation of 32 equations in A_{im} and B_{in} from the boundary conditions by taking four terms in the series ($m, n = 1, 3, 5$ and 7) were carried out on a desk calculator. The solution of these equations and the computations of stresses and moments were done on an IBM 1620 digital computer.

The results are presented in the form of contours in *Fig* 3.

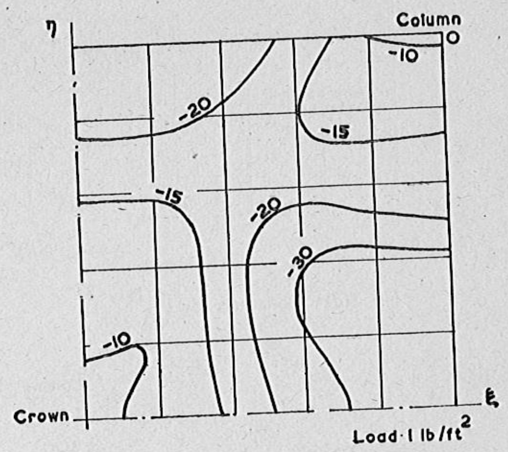

Fig 3(*a*) Contours for $\left(0{\cdot}513 \times 10^6 \frac{W}{a}\right)$, positive upwards

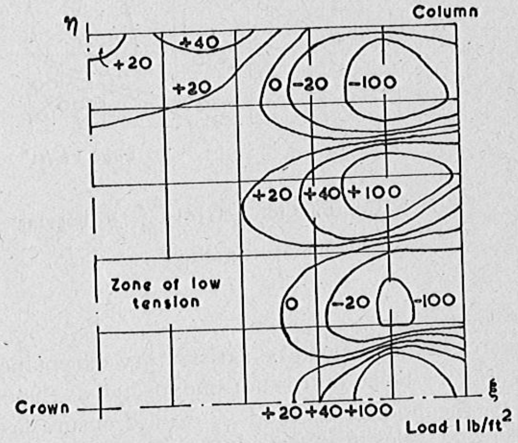

Fig 3(*b*) Contours for $\left(\frac{Nx}{2{\cdot}34h}\right)$ lb/in²

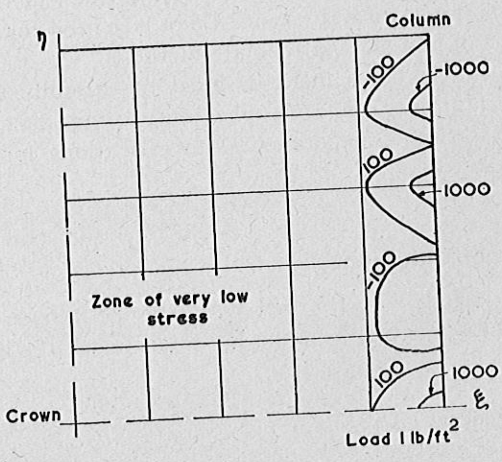

Fig 3(*c*) Contours for $\left(\frac{Ny}{2{\cdot}34h}\right)$ lb/in²

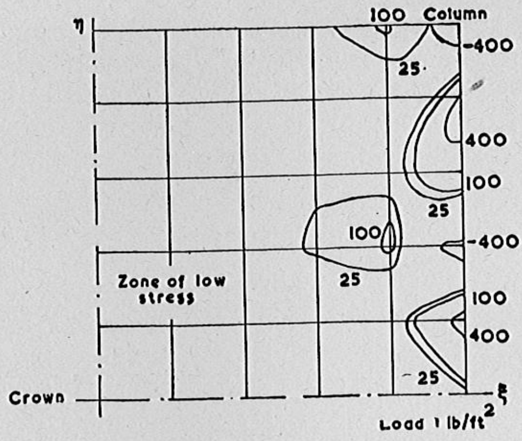

Fig 3(d) Contours for $\left(\frac{Nxy}{2\cdot 34h}\right)$

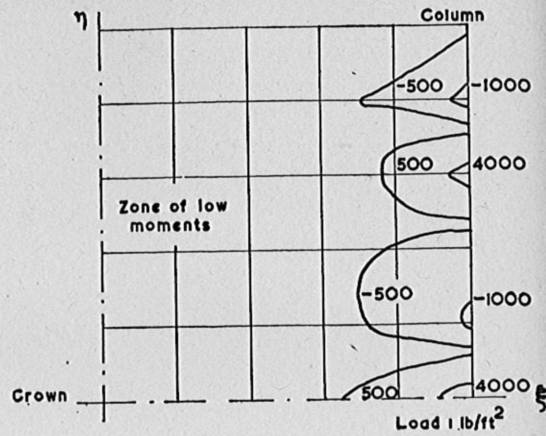

Fig 3(f) Contours for $\left(4\cdot 161\ \frac{My}{h^2}\right)$, hogging moments positive

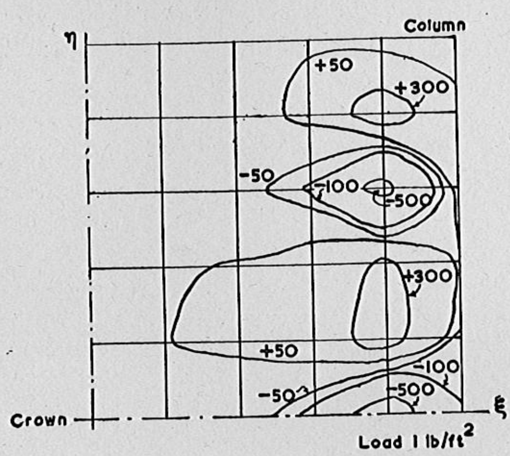

Fig 3(e) Contours for $\left(4\cdot 161\ \frac{Mx}{h^2}\right)$, hogging moments positive

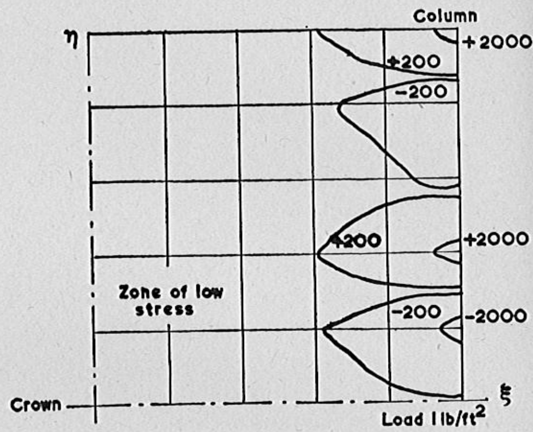

Fig 3(g) Contours for $\left(4\cdot 161\ \frac{Mxy}{h^2}\right)$

Discussion :

The results appear to be satisfactory except near the edges. Such a behaviour is not unexpected as the method used — the Fourier method — cannot ensure fast convergence of series near the edges.

The determination of stresses and moments in a doubly-curved shell supported on edge beams is a complex problem from the point of view of mathematical solution as well as that of computation. It may not be possible for a design engineer to use exact methods like the Fourier analysis, hence there is a need for development of reliable approximate methods.

However, such methods must be carefully tested to ascertain their range of applicability, using exact methods or a series of experimental investigations for various parameters.

Acknowledgment

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APPENDIX I

Non-dimensional boundary equations

Notation :

$\bar{A}_{tm} = A_{tm}\, m^8/a^5$

$\bar{B}_{tn} = B_{tn}\, n^8/a^5$

$\bar{\Upsilon}_{tm} = \Upsilon_{tm}\, b/m\pi$

$\bar{\lambda}_{tn} = \lambda_{tn}\, a/n\pi$

$\bar{k}_n = k_n\, n^8/a^5$

$K_1 = b/a$	$K_2 = \pi/2K_1$
$K_3 = ar$	$K_4 = at$
$K_5 = h/a$	$K_{10} = C/Ea^4$
$K_{11} = Eh\,a^2/D$	$K_{12} = I_z/a^4$
$K_{13} = I_y/a^4$	$K_{14} = A_o/a^2$
$\bar{K}_{10} = \bar{C}/E\,b^4$	$\bar{K}_{12} = \bar{I}_z/b^4$
$\bar{K}_{13} = \bar{I}_y/b^4$	$\bar{K}_{14} = A_o/b^2$

Edge $y = b/2$

The following non-dimensional equations correspond to the boundary equations 6 (*a*, *b*, *c*, *d*), respectively.

$$\sum_{i=1,2,3,4}\left[\sum_{m=1,3,..}\bar{A}_{im}\frac{\pi^4K_5^3}{3(1-\nu^2)K_1^7}\frac{n}{m}\frac{\sin\frac{n\pi}{2}\sin\frac{m\pi}{2}}{\left(n^2+\frac{m^2}{K_1^2}\bar{\gamma}_{im}^2\right)}\right.$$

$$\times(\gamma_{im}^2-1)^2\left\{(2-\nu)\bar{\gamma}_{im}^2-1\right\}\cosh mK_2\bar{\gamma}_{im}$$

$$+\sum_{m=1,3..}\bar{A}_{im}\frac{2K_5K_3^2}{K_1^4}\frac{n}{m^4}\frac{\sin\frac{n\pi}{2}\sin\frac{m\pi}{2}}{\left(n^2+\frac{m^2}{K_1^2}\bar{\gamma}_{im}^2\right)^2}\bar{\gamma}_{im}\left(1-\frac{t}{r}\bar{\gamma}_{im}^2\right)$$

$$\times\left\{\left(n^2+\frac{m^2}{K_1^2}\bar{\gamma}_{im}^2\right)\pi\tanh mK_2\bar{\gamma}_{im}-\frac{4m\bar{\gamma}_{im}}{K_1}\right\}\cosh mK_2\gamma_{im}$$

$$-\bar{B}_{in}\frac{\pi^2}{n^4}\left[-K_{13}n^4\pi^4\left(\bar{\lambda}_{in}^2-1\right)^2-\frac{K_4^2K_1K_5}{2}\left(1-\frac{r}{t}\bar{\lambda}_{in}^2\right)\right]$$

$$\times\cosh\frac{n\pi K_1}{2}\bar{\lambda}_{in}+\bar{B}_{in}\frac{\pi^5K_5^3}{12(1-\nu^2)}\frac{1}{n}\bar{\lambda}_{in}\left(\bar{\lambda}_{in}^2-1\right)^2$$

$$\times\left\{(2-\nu)-\bar{\lambda}_{in}^2\right\}\sinh\frac{n\pi k_1}{2}\bar{\lambda}_{in}$$

$$+\sum_{m=1,3..}^{m\neq n}\bar{B}_{im}\frac{2\pi(-1)^{\frac{m+n}{2}}n}{m^4(m^2-n^2)}K_3K_4K_5\bar{\lambda}_{im}\left(1-\frac{r}{t}\bar{\lambda}_{im}^2\right)$$

$$\times\sinh\frac{m\pi K_1}{2}\bar{\lambda}_{im}-\bar{B}_{in}\frac{\pi(-1)^n}{2n^5}K_3K_4K_5\bar{\lambda}_{in}$$

$$\left.\times\left(1-\frac{r}{t}\bar{\lambda}_{in}^2\right)\sinh\frac{n\pi K_1}{2}\bar{\lambda}_{in}\right]$$

$$-\bar{k}_n\frac{\pi^2}{n^4}\left(K_{13}n^4\pi^4+K_5K_4^2\frac{K_1}{2}\right)=0 \quad\ldots\ldots\ldots\ldots(A1)$$

$$\sum_{i=1,2,3,4}\left|\sum_{m=1,3,..}\bar{A}_{im}\frac{4\pi^5}{K_1^7}\frac{n}{m}\frac{\sin\frac{n\pi}{2}\sin\frac{m\pi}{2}}{\left(n^2+\frac{m^2\bar{\gamma}_{im}^2}{K_1^2}\right)}\right.$$

$$\left[K_{12}K_3\bar{\gamma}_{im}^2\times\left(1-\frac{t}{r}\bar{\gamma}_{im}^2\right)\left\{1-(2+\nu)\bar{\gamma}_{im}^2\right\}\right.$$

$$+(\bar{\gamma}_{im}^2-1)^2\left[\frac{K_4K_1K_5^3}{24(1-\nu^2)}\left\{(2-\nu)\bar{\gamma}_{im}^2-1\right\}\right.$$

$$\left.\left.-K_{12}K_3\bar{\gamma}_{im}^2\right]\right]\cosh mK_2\bar{\gamma}_{im}+\bar{B}_{in}\pi^7\left[\frac{K_{12}K_4K_1}{2}\right.$$

$$\left.\left(\bar{\lambda}_{in}^2-1\right)^2+\frac{K_5K_4}{\pi^4n^4}\left(1-\frac{r}{t}\lambda_{in}^2\right)\right]\times\cosh\frac{n\pi K_1}{2}\bar{\lambda}_{in}+\bar{B}_{in}$$

$$\frac{\pi^6}{n}\bar{\lambda}_{in}\left[K_{12}K_4\left(1-\frac{r}{t}\bar{\lambda}_{in}^2\right)\times\left\{(2+\nu)-\bar{\lambda}_{in}^2\right\}\right.$$

$$\left.+\left(\bar{\lambda}_{in}^2-1\right)\left\{\frac{K_5^3K_1K_4}{24(1-\nu^2)}\left(2-\nu-\bar{\lambda}_{in}^2\right)-K_3K_{12}\right\}\right]$$

$$\left.\sinh\frac{n\pi K_1}{2}\bar{\lambda}_{in}\right|$$

$$+\bar{k}_n\frac{\pi^4K_4}{n^4}\left(K_{12}K_1\frac{\pi^4n^4}{2}+K_5\right)=0\quad\ldots\ldots\ldots\ldots(A2)$$

$$\sum_{i=1,2,3,4}\left[\sum_{m=1,3..}\bar{A}_{im}\frac{4K_{10}}{K_1^7}\frac{n}{m}\left(\bar{\gamma}_{in}^2-1\right)^2\bar{\gamma}_{im}^2\frac{\sin\frac{n\pi}{2}\sin\frac{m\pi}{2}}{\left(n^2+\frac{m^2}{K_1^2}\right)}\right.$$

$$\times\cosh mK_2\bar{\gamma}_{im}+\bar{B}_{in}\frac{K_5^3}{12(1-\nu^2)}\frac{1}{n^2}\left(\bar{\lambda}_{in}^2-1\right)^2$$

$$\times\left(\bar{\lambda}_{in}^2-\nu\right)\cosh\frac{n\pi K_1}{2}\bar{\lambda}_{in}-\bar{B}_{in}\frac{\pi K_{10}}{n}\bar{\lambda}_{in}$$

$$\left.\times\left(\bar{\lambda}_{in}^2-1\right)^2\sinh\frac{n\pi K_1}{2}\bar{\lambda}_{in}\right]$$

$$-\bar{k}_n\frac{\nu K_5^3}{12(1-\nu^2)}\frac{1}{n^2}=0\quad\ldots\ldots\ldots\ldots(A3)$$

$$\sum_{i=1,2,3,4}\left[-\sum_{m=1,3,..}\bar{A}_{im}\frac{4K_5K_3}{K_1^3}\frac{n}{m^5}\frac{\sin\frac{n\pi}{2}\sin\frac{m\pi}{2}}{\left(n^2+\frac{m^2\bar{\gamma}_{im}^2}{K_1^2}\right)}\right.$$

$$\times\left(1-\frac{t}{r}\bar{\gamma}_{im}^2\right)\cosh mK_2\bar{\gamma}_{im}+\bar{B}_{in}\frac{\pi^2K_{14}K_4}{n^4}$$

$$\times\left(\bar{\lambda}_{in}^2+\nu\right)\left(1-\frac{r}{t}\bar{\lambda}_{in}^2\right)\cosh\frac{n\pi K_1}{2}\bar{\lambda}_{in}$$

$$+\bar{B}_{in}\frac{\pi K_5K_4}{n^5}\bar{\lambda}_{in}\left(1-\frac{r}{t}\bar{\lambda}_{in}^2\right)\sinh\frac{n\pi K_1}{2}\bar{\lambda}_{in}$$

$$+\sum_{m=1,3,..}^{m\neq n}\bar{B}_{im}\frac{2\pi^2K_{14}K_3}{m^3}\frac{n(-1)^{\frac{m+n}{2}}}{(m^2-n^2)}\left(\bar{\lambda}_{im}^2-1\right)^2\cosh\frac{m\pi K_1}{2}\bar{\lambda}_{im}$$

$$-\bar{B}_{in}\pi^2K_{14}K_3\frac{(-1)^n}{2n^4}\left(\bar{\lambda}_{in}^2-1\right)^2\cosh\frac{n\pi K_1}{2}\bar{\lambda}_{in}$$

$$+\sum_{m=1,3,..}\bar{A}_{im}\frac{4K_5K_3}{K_1^3}\frac{1}{m^5n}\left(1-\frac{t}{r}\bar{\gamma}_{im}^2\right)\sin\frac{n\pi}{2}\sin\frac{m\pi}{2}$$

$$\left.\times\cosh mK_2\bar{\gamma}_{im}\right]+\bar{k}_n\frac{\pi^2K_{14}}{n^4}\left(\nu K_4-\frac{(-1)^nK_3}{2}\right)$$

$$+\sum_{m=1,3,..}^{m\neq n}\bar{k}_m 2\pi^2K_{14}K_3\frac{n(-1)^{\frac{m+n}{2}}}{m^3(m^2-n^2)}=0\quad\ldots\ldots(A4)$$

The equations for the edge $x = a/2$ can be written from the preceding equations by interchanging x and y, A_{im} and B_{in}, and m and n. The various parameters are modified as follows (the arrow stands for ' is replaced by ') :

$K_1 \rightarrow 1/K_1$		$K_2 \rightarrow K_2 K_1^2$	
$K_3 \rightarrow K_4 K_1$		$K_4 \rightarrow K_3 K_1$	
$K_5 \rightarrow K_5/K_1$		$K_{11} \rightarrow K_{11} K_1^2$	
$K_{10} \rightarrow \bar{K}_{10}$		$K_{12} \rightarrow \bar{K}_{12}$	
$K_{13} \rightarrow \bar{K}_{13}$		$K_{14} \rightarrow \bar{K}_{14}$	

The load terms need special treatment. Following the procedure outlined for the edge $y = b/2$, the load terms for $x = a/2$ are found to be,

for equation $A1$:

$$-\sum_{n=1,3,..} \bar{k}_n K_5^3 K_1^4 \frac{\pi^4}{3(1-\nu^2)} \frac{\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn}$$

for equation $(A2)$:

$$-\sum_{n=1,3,..} \bar{k}_n K_1^4 K_3 K_5^3 \frac{\pi^5}{6(1-\nu^2)} \frac{\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn}$$

for equation $(A3)$: zero

for equation $(A4)$:

$$-\sum_{n=1,3,..} \bar{k}_n 4 K_1^3 K_4 K_5 \frac{\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{n^5 m}$$

APPENDIX II

Fourier Transforms

Expansions for $-a/2 \leqslant x \leqslant +a/2$

(1) $$\cosh\gamma x = \sum_{l=1,3,..} \frac{4l\pi \sin\frac{l\pi}{2}}{(l^2\pi^2+\gamma^2a^2)} \cosh\frac{\gamma a}{2} \cos\frac{l\pi x}{a}$$

(2) $$x\sin\frac{n\pi x}{a} = \sum_{l=1,3,..}^{l\neq n} \frac{2al(-1)^{\frac{n+l}{2}}}{\pi(n^2-l^2)}\cos\frac{l\pi x}{a} - \frac{a(-1)^n}{2n\pi}\cos\frac{n\pi x}{a}$$

(3) $$x\sinh\gamma x = \sum_{l=1,3,..} \frac{2al\pi(-1)^{\frac{l-1}{2}}}{(a^2\gamma^2+l^2\pi^2)^2}\cosh\frac{\gamma a}{2} \times \left[(a^2\gamma^2+l^2\pi^2)+\tanh\frac{\gamma a}{2} - 4a\gamma\right]\cos\frac{l\pi x}{a}$$

(4) $$1 = \sum_{l=1,3,..} \frac{4(-1)^{\frac{l-1}{2}}}{l\pi}\cos\frac{l\pi x}{a}$$

In cases (1), (2), and (3), in order to get an expansion in terms of only odd values of l, the function is expanded as shown in *Fig* 4.

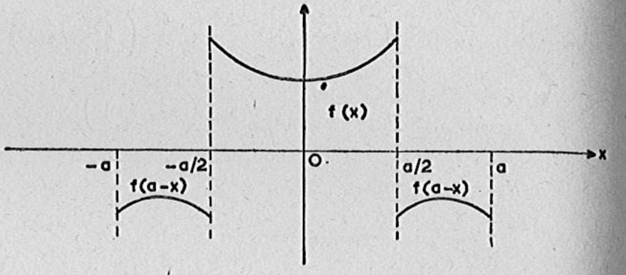

Fig 4 Assumed distribution of function for Fourier transformation

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